

ON THE SMALL MEASURE EXPANSION PHENOMENON IN CONNECTED NONCOMPACT NONABELIAN GROUPS

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ABSTRACT. Suppose G is a connected noncompact locally compact group, A, B are nonempty and compact subsets of G , μ is a left Haar measure on G . Assuming that G is unimodular, and $\mu(A^2) < K\mu(A)$ with $K > 1$ a fixed constant, our first result shows that there is a continuous surjective group homomorphism $\chi : G \rightarrow L$ with compact kernel, where L is a Lie group with

$$\dim(L) \leq \lfloor \log K \rfloor (\lfloor \log K \rfloor + 1)/2.$$

We also demonstrate that this dimension bound is sharp, establish the relationship between A and its image under the quotient map, and obtain a more general version of this result for the product set AB without assuming unimodularity.

Our second result classifies G, A, B where A, B have nearly minimal expansions (when G is unimodular, this just means $\mu(AB)$ is close to $\mu(A) + \mu(B)$). This completes the remaining noncompact cases of the Kemperman inverse problem and answers a question suggested by Griesmer and Tao.

The proofs of both results involve a new analysis of locally compact group G with bounded $n - h$, where $n - h$ is an invariant of G appearing in the recently developed nonabelian Brunn–Minkowski inequality. We also generalize Ruzsa’s distance and related results to possibly nonunimodular locally compact groups.

1. INTRODUCTION

A central problem in additive combinatorics is to obtain descriptions of a subset A of an abelian group G such that $A + A := \{a_1 + a_2 : a_1, a_2 \in A\}$ has small expansion (i.e., $|A + A| < K|A|$ for a fixed constant K). Freiman-type results tell us that such A must be commensurable to a coset progression; the theorem for abelian groups was proven by Green and Ruzsa [7], and the quantitative bound was later

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improved by Sanders [22]. When $K = 2$, a stronger conclusion in the same line is given by Kemperman's structure theorem [16], which classifies finite subsets A, B of an abelian group G such that $|A + B| < |A| + |B|$. Another famous result in this direction is Freiman's $(3k - 4)$ theorem, which tells us that if $A \subseteq \mathbb{Z}$ satisfies $|A + A| \leq 3|A| - 4$, then A must be contained in an arithmetic progression of length $2|A| - 1$. The theory have been extended to various settings, which brings together ideas from different areas of mathematics. In this paper, we obtain the counterpart of the above results for connected noncompact locally compact groups.

For the rest of the introduction, let G be a *connected* locally compact group, μ_G a left Haar measure on G , and $A, B \subseteq G$ are nonempty and compact. As usual, we let $AB = \{ab : a \in A, b \in B\}$ be the productset of A and B . We also write A^2 when $A = B$. Even though we have a more general result that applies arbitrary G , we will assume for simplicity in the next result that G is unimodular (i.e., μ_G is also a right Haar measure on G , which holds, for example, when G is a semisimple Lie group). Our first main theorem tells us that small expansion in an arbitrary locally compact group arise from small expansion in a Lie group of small dimension:

Theorem 1.1 (Small measure expansions in unimodular noncompact groups). *Let G be a connected unimodular noncompact locally compact group, μ_G is a left Haar measure. Suppose A is a compact subset of G of positive measure, and*

$$(1) \quad \mu_G(A^2) < K\mu_G(A).$$

Then there is a continuous surjective group homomorphism $\chi : G \rightarrow L$ with compact kernel, where L is a connected Lie group with

$$\dim(L) \leq \lfloor \log K \rfloor (\lfloor \log K \rfloor + 1) / 2.$$

Moreover, with $A' = \chi(A)$, we have $\mu_L((A')^2) \leq 64K^6\mu_L(A')$.

The dimension bound in the above theorem is sharp. For example, let G be the identity component $\mathrm{SO}^+(r, 1)$ of the pseudo-orthogonal group $\mathrm{SO}(r, 1)$. In this case, $\dim(G) = r(r+1)/2$. The group G has $\mathrm{SO}(r)$ as a maximal compact subgroup, which has dimension $r(r-1)/2$. For every $\varepsilon > 0$, by taking a small open neighborhood A of $\mathrm{SO}(r)$ in $\mathrm{SO}^+(r, 1)$, the computation by the last three authors in [15] gives us $\mu_G(A^2) < 2^{r+\varepsilon}\mu_G(A)$.

Theorem 1.1 has the following immediate corollary, which can be applied to works by Hrushovski and Rodriguez Fanlo [11], Rodriguez Fanlo [20], and Machado [19] on variants of small expansion problems.

Corollary 1.2. *Let G be a connected unimodular noncompact Lie group with no nontrivial compact normal subgroups. Suppose there is a compact subset $A \subseteq G$ of positive measure, and $\mu_G(A^2) < K\mu_G(A)$. Then $\dim(G) \leq \lfloor \log K \rfloor (\lfloor \log K \rfloor + 1) / 2$.*

An ineffective version of Theorem 1.1, with no quantitative bound on the dimension of L can be deduced from Carolino’s classification of measure approximate groups of locally compact groups [3]. When G is a nilpotent Lie group with no compact normal subgroups, the bound in Corollary 1.2 can be strengthened to $\lfloor \log K \rfloor$. This is a result by Gelander and Hrushovski [10] and they used it to characterize finite approximate groups; see also [23].

A key ingredient of the proof of Theorem 1.1 is the recently developed nonabelian Brunn–Minkowski inequality [15] by the last three authors. This produces a structural constraint $n - h < \log K + 1$ on the group G , where $n - h$ is an invariant of G roughly measuring its “noncompact dimension”. A main new component of this paper is a structural study of G with $n - h < \log K + 1$, which yields the desired χ and L . Even though the Brunn–Minkowski inequality obtained in [15] is sharp for a large class of groups (e.g. matrix groups, solvable groups), it can be nonsharp for certain other groups. Somewhat surprisingly, we still manage to deduce a sharp dimension bound for all locally compact groups. This is done through a more involved analysis. The small expansion condition of A' in Theorem 1.1 can be obtained using a nonunimodular generalization of an argument of Tao [24] to replace A with a commensurable approximate groups, as well as a geometric spillover argument.

Theorem 1.1 will be presented as a special case of Theorem 7.1, which applies to all connected noncompact locally compact groups and pairs (A, B) with the product set AB have small measures in an appropriate sense compared to the measures of A and B . This requires us to introduce in addition an notion of Ruzsa’s distance for nonunimodular groups and develop the related machinery.

In the rest of the introduction, we further explore the behaviours of sets and groups in Theorem 7.1 when a measure expansion strictly less than 4 occurs.

Theorem 1.3 (Nearly minimal expansions in noncompact groups). *Let G be a connected noncompact locally compact group, μ_G a left Haar measure, and ν_G a right Haar measure on G . Suppose A, B are compact subsets of G , each of positive measure, and*

$$(2) \quad \left(\frac{\nu_G(A)}{\nu_G(AB)} \right)^{\frac{1}{2}} + \left(\frac{\mu_G(B)}{\mu_G(AB)} \right)^{\frac{1}{2}} > 1,$$

Then G is unimodular. In particular, (2) is just $\mu_G(AB)^{1/2} < \mu_G(A)^{1/2} + \mu_G(B)^{1/2}$, which is $\mu_G(A^2) < 4\mu_G(A)$ when $A = B$.

Moreover, we have the following:

- (i) *If $\mu_G(AB) < \mu_G(A) + \mu_G(B) + \frac{1}{20} \min\{\mu_G(A), \mu_G(B)\}$, then there is a continuous surjective group homomorphism $\chi : G \rightarrow \mathbb{R}$ with compact kernel, and two intervals $I, J \subseteq \mathbb{R}$, such that $A \subseteq \chi^{-1}(I)$ and $B \subseteq \chi^{-1}(J)$, and*

$$\mu_G(\chi^{-1}(I)) - \mu_G(A) < 100\mathfrak{d}_G(A, B) \quad \mu_G(\chi^{-1}(J)) - \mu_G(B) < 100\mathfrak{d}_G(A, B)$$

- where we set $\mathfrak{d}_G(A, B) = \mu_G(AB) - \mu_G(A) - \mu_G(B)$.
- (ii) If $\mu_G(AB) = \mu_G(A) + \mu_G(B)$, then there is a continuous surjective group homomorphism $\chi : G \rightarrow \mathbb{R}$ with compact kernel, and two intervals $I, J \subseteq \mathbb{R}$, such that $A = \chi^{-1}(I)$ and $B = \chi^{-1}(J)$.

We conjecture that one can replace $1/20$ in Theorem 1.3 by 1 . It is easy to see that replacing $1/20$ by a constant > 1 , the structural characterization obtained in statement (i) will no longer hold. With more efforts, we also extend Theorem 1.3 to sets with measure expansions at most 4 , see Theorem 7.3.

Theorem 1.3 is the solution to noncompact cases of the Kemperman inverse problem, which we will now describe. For any compact sets A, B in a connected unimodular group G , Kemperman [17] proved in 1964 that

$$\mu_G(AB) \geq \min\{\mu_G(A) + \mu_G(B), \mu_G(G)\}.$$

The following more general inequality for possibly nonunimodular G was also found by Kemperman in the same paper:

$$\min \left\{ \frac{\nu_G(A)}{\nu_G(AB)} + \frac{\mu_G(B)}{\mu_G(AB)}, \frac{\mu_G(G)}{\mu_G(AB)} \right\} \leq 1.$$

where μ_G and ν_G are the left and the right Haar measure of G . Recently, the second and the third authors [13] generalize the above inequality to possibly disconnected locally compact groups.

The Kemperman inverse problem is to investigate the structural characterizations when the equality in Kemperman's inequality happens or nearly happens. The question for equality was asked by Kemperman himself in [17]. By "equality nearly happens", we mean that when there is $\delta > 0$ such that

$$(3) \quad \mu_G(AB) \leq \mu_G(A) + \mu_G(B) + \delta,$$

and we aim to obtain classifications of the unimodular group G and compact sets A, B . This problem was proposed by Griesmer and Tao [25].

Prior to Theorem 1.3, the answers to the Kemperman inverse problem is only known for compact G and when G is $(\mathbb{R}, +)$. In more details, with the further assumption that G is abelian and compact, the conditions for equality to happen were determined by Kneser [18], and for equality to nearly happens by Tao [25]. Subsequently, Christ and Iliopoulou [4] obtained a sharp exponent version of Tao's result, and Griesmer [8] generalizes this to disconnected and compact abelian G . Recently, the second and the third authors [14] determined when equality holds in the Kemperman inequality for nonabelian and unimodular G . They also characterized G, A, B when the equality nearly happens under the further assumption that G is compact, but their methods does not work when G is noncompact. For $G = (\mathbb{R}, +)$,

the answers follow from the folkloric solution to the 1-dimensional Brunn–Minkowski inverse problem.

To prove Theorem 1.3, one need Theorem 7.1 (the assymmetric nonunimodular version of 1.1) as the new ingredient together with some further geometric arguments. On the other hand, we note that the method developed in this paper does not work for compact groups, as the Brunn–Minkowski inequality becomes trivial when the group is compact.

Organization of the paper. The paper is organized as follows. In Section 2, we provide some background on locally compact groups. In Section 3, we introduce the Brunn–Minkowski coefficient of groups, and prove a bounded dimension Lie quotient theorem for groups with bounded Brunn–Minkowski coefficients. In Section 4, we introduce a nonunimodular version of the Ruzsa distance, and use it to link the small expansion sets and open approximate groups. In Section 5, we prove quotient domination results when the sets have small measure expansions. In Section 6, we study sets with minimal expansions in nonunimodular groups. We then prove all the main theorems in Section 7.

2. PRELIMINARIES

Throughout the paper, G is a connected locally compact group, μ_G is a left Haar measure and ν_G is a right Haar measure on G . When the ambient group is clear in the context, we sometimes simply use μ and ν to denote the left and the right Haar measures respectively. All the log in this paper are logarithms with base 2. We say G is **unimodular** if a left Haar measure is also a right Haar measure on G .

The structure of locally compact groups was characterized by Gleason [6] and Yamabe [26], which resolved Hilbert’s 5th problem:

Fact 2.1 (Gleason–Yamabe Theorem). *Suppose G is a locally compact group. Then there is an open subgroup G' of G , such that for any open neighborhood U of G' containing identity, there is a compact normal subgroup $H \subseteq U$, such that G'/H is a Lie group.*

Given a locally compact group G , and μ is a left Haar measure on G . Define $\mu_x(A) = \mu(Ax)$ for every $x \in G$ and every measurable sets A . For every $x \in G$, the modular function is defined by $\Delta_G : x \mapsto \mu_x/\mu$. In particular when the image of Δ_G is always 1, G is unimodular. In general, Δ_G takes values in $\mathbb{R}^{>0}$. The following fact from [1] records some basic properties of the modular function:

Fact 2.2. *Let G be a locally compact group, μ a left Haar measure and ν a right Haar measure on G .*

- (i) *Suppose H is a normal closed subgroup of G , then $\Delta_H = \Delta_G|_H$. In particular, if $H = \ker \Delta_G$, then H is unimodular.*

- (ii) The function $\Delta_G : G \rightarrow \mathbb{R}^{>0}$ is a continuous homomorphism.
- (iii) For every $x \in G$ and every measurable set A , we have $\mu(Ax) = \Delta_G(x)\mu(A)$, and $\nu(xA) = \Delta_G^{-1}(x)\nu(A)$.
- (iv) There is a constant c such that $\int_G f \, d\mu = c \int_G f \Delta_G \, d\nu$ for every $f \in C_c(G)$.

Let ϕ be an automorphism on G . Clearly, $\phi^{-1}(\mu)$ is also a left Haar measure on G . The number $\phi^{-1}(\mu)/\mu$ is called **modulus of the automorphism** ϕ , and is denoted by $\text{mod}_G(\phi)$. We use the following fact from [2, Proposition 2.7.10]

Fact 2.3. *Let G be a locally compact group, H be a closed normal subgroup of G , and $\pi : G \rightarrow G/H$ is the quotient map. Let u be an automorphism of G with $u(H) = H$, u' is the restriction of u on H , and u'' is the projection of u on G/H . Then*

$$\text{mod}_G(u) = \text{mod}_H(u') \cdot \text{mod}_{G/H}(u'').$$

We use the following integral formula [5, Theorem 2.49] in our proofs.

Fact 2.4 (Quotient integral formula). *Let G be a locally compact group, and let H be a closed normal subgroup of G . Given μ_G, μ_H left Haar measures on G and on H . Then there is a unique left Haar measure $\mu_{G/H}$ on G/H , such that for every $f \in C_c(G)$,*

$$\int_G f(x) \, d\mu_G(x) = \int_{G/H} \int_H f(xh) \, d\mu_H(h) \, d\mu_{G/H}(x).$$

The following well-known fact is the Freiman $3k - 4$ theorem over \mathbb{R} .

Fact 2.5 ($3k - 4$ theorem in \mathbb{R}). *Suppose A, B are compact subsets of \mathbb{R} , and λ is the Lebesgue measure on \mathbb{R} . If*

$$\lambda(A + B) < \lambda(A) + \lambda(B) + \min\{\lambda(A), \lambda(B)\},$$

then there are compact intervals $I, J \subseteq \mathbb{R}$, with

$$\lambda(I) \leq \lambda(A + B) - \lambda(B), \quad \lambda(J) \leq \lambda(A + B) - \lambda(A),$$

and $A \subseteq I, B \subseteq J$.

3. SMALL EXPANSIONS AND LOW DIMENSION LIE QUOTIENTS

Let G be a noncompact group. Given two compact sets $A, B \subseteq G$ with positive measures, the **Brunn–Minkowski coefficient** of (A, B) is the real number $r \geq 0$ such that

$$\left(\frac{\nu(A)}{\nu(AB)} \right)^{\frac{1}{r}} + \left(\frac{\mu(B)}{\mu(AB)} \right)^{\frac{1}{r}} = 1.$$

In this case, we write $\text{BM}(A, B) = r$. Assuming that G is unimodular, we have

$$\mu(AB)^{1/r} = \mu(A)^{1/r} + \mu(B)^{1/r}.$$

If, moreover, $A = B$, then $\mu(A^2) = 2^r \mu(A)$. Thus, we can think of r as a measuring of the expansion of the pair (A, B) where we allow A and B to have different size.

The **Brunn–Minkowski coefficient** of G is the largest nonnegative real number r such that

$$\text{BM}(A, B) \geq r \quad \text{for every compact } A, B \subseteq G \text{ with positive measures.}$$

In this case, we write $\text{BM}(G) = r$. The classical Brunn–Minkowski inequality for \mathbb{R}^d tells us that $\text{BM}(\mathbb{R}^d) = d$.

The problem of determining the Brunn–Minkowski for general locally compact group was studied by the second, third, and fourth author in [15]. Toward stating our conjectural answer, we define the **noncompact Lie dimension** $\text{ndim}(G)$ of G as follows. Using Theorem 2.1, one can choose an open subgroup G' of G and a normal compact subgroup H of G' such that G'/H is a Lie group. Set

$$\text{ndim}(G) = \dim(G'/H) - \max\{\dim(K) : K \text{ is a compact subgroup of } G'/H\}.$$

It was verified in [15] that $\text{ndim}(G)$ is well-defined (i.e., it is independent of the choice of G'). The following properties of $\text{ndim}(G)$, proven in [15], will be used later on.

Fact 3.1. *Let G be a locally compact group. Then*

- (i) *If G is compact, then $\text{ndim}(G) = 0$.*
- (ii) *Suppose $1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1$ is a short exact sequence of connected Lie groups. Then $\text{ndim}(G) = \text{ndim}(H) + \text{ndim}(G/H)$.*

Below is the conjecture proposed in [15]; it was also proven there that it suffices to verify the conjecture for simply connected simple Lie groups.

Conjecture 3.2 (Nonabelian Brunn–Minkowski Conjecture). *If G is a locally compact group with noncompact Lie dimension n , then $\text{BM}(G) = n$.*

We next review the result toward Conjecture 3.2 proven in [15]; this will be a central ingredient in our later proof. We have the following fact of Lie groups, which is a consequence of [9, Proposition 5.4.3]:

Fact 3.3. *Suppose \mathfrak{g} is a Lie algebra. Then \mathfrak{g} has a largest solvable ideal \mathfrak{q} . If G is a Lie group with Lie algebra \mathfrak{g} and $\exp : \mathfrak{g} \rightarrow G$ is the exponential map, then $Q = \langle \exp(\mathfrak{q}) \rangle$ is the largest closed connected solvable normal subgroup of G . Hence, Q is a characteristic subgroup of G .*

We call such Q in Fact 3.3 the *radical* of G . A Lie group is *semisimple* if its Lie algebra is semisimple, or equivalently, if it has trivial radical. The following fact is also a consequence of [9, Proposition 5.4.3]:

Fact 3.4. *Let G be a connected Lie group. Let Q be the radical of G . Then $S = G/Q$ is a semisimple Lie group.*

Now we are going to define the **helix dimension** of a locally compact group G , denoted by $\text{hdim}(G)$. Let G'/H be the Lie group obtained from Theorem 2.1, and let Q be the radical of G'/H . Note that $(G'/H)_0/Q$ has discrete center $Z((G'/H)_0/Q)$. We set

$$\text{hdim}(G) = \text{rank}(Z((G'/H)_0/Q)).$$

In [15], $\text{hdim}(G)$ is well-defined, i.e., it is independent of the choice of G' . In general, as shown in [15], $\text{hdim}(G)$ is not split with respect to short exact sequences. However, we have the following fact which is proven in [15].

Fact 3.5. *Let G be a locally compact group. Then*

- (i) *Suppose $1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1$ is a short exact sequence of connected locally compact groups, and H is compact. Then $\text{hdim}(G) = \text{hdim}(G/H)$.*
- (ii) *Suppose $1 \rightarrow H \rightarrow G \rightarrow \mathbb{R} \rightarrow 1$ is a short exact sequence of locally compact groups. Then $\text{hdim}(G) = \text{hdim}(H)$.*
- (iii) *Let G be a connected Lie group of noncompact dimension n and helix dimension h . Then $n \geq 3h$. If G is semisimple, and $n > 0$, then $n \geq 2$.*

The following fact is the main result of [15].

Fact 3.6 (Jing–Tran–Zhang). *Let G be a locally compact group such that $\text{ndim}(G) = n$ and $\text{hdim}(G) = h$. Then,*

$$n - h \leq \text{BM}(G) \leq n.$$

Moreover, solvable locally compact groups and real algebraic groups has helix dimension 0, so this verifies the nonabelian Brunn–Minkowski conjecture for these groups.

We will also need the following facts about Lie groups.

Fact 3.7. *We have the following:*

- (i) *Any two maximal compact subgroups of a connected Lie group are conjugate.*
- (ii) *A maximal compact subgroup of $\text{GL}(n, \mathbb{R})$ is the orthogonal group $O(n)$, which has dimension $n(n-1)/2$.*

We now prove the following key lemma.

Lemma 3.8. *Let G be a connected locally compact group with noncompact dimension n and helix dimension h . Then G has a compact connected characteristic subgroup H such that G/H is a Lie group of dimension at most $(n-h)(n-h+1)/2$.*

Proof. By the Gleason–Yamabe theorem (Fact 2.1), we can assume that G is a connected Lie group. Let us divide the proof into steps.

Step 1. We first prove the weaker statement that there is a compact connected characteristic subgroup H of G such that $\dim G/H \leq n(n+1)/2$. Let K be a

maximal compact subgroup of G , and denote the Lie algebras of G and K by \mathfrak{g} and \mathfrak{k} , respectively. Let $\text{Ad}_K : K \rightarrow \text{GL}(\mathfrak{g})$ be the adjoint representation of K on \mathfrak{g} . Since K is compact, there is an Ad_K -invariant inner product on \mathfrak{g} , and the orthogonal complement of \mathfrak{k} in \mathfrak{g} , denoted by \mathfrak{p} , is Ad_K -invariant. Let

$$\rho : K \rightarrow \text{GL}(\mathfrak{p})$$

be the subrepresentation of Ad_K , and $H := \text{Ker}(\rho)^\circ$ be the identity component of the $\text{Ker}(\rho)$. We will show that the compact group H satisfies the requirement.

First, the space \mathfrak{p} has dimension n , so by Fact 3.7(ii), we have

$$\dim G/H = \dim G/K + \dim K/H = n + \dim \rho(K) \leq n + n(n-1)/2 = n(n+1)/2.$$

Next, we prove that H is normal in G . Since H is connected, it suffices to show that the Lie algebra \mathfrak{h} of H is an ideal of \mathfrak{g} . Clearly, \mathfrak{h} is an ideal of \mathfrak{k} . On the other hand, since H acts trivially on \mathfrak{p} , we have $[\mathfrak{h}, \mathfrak{p}] = 0$. Thus $[\mathfrak{h}, \mathfrak{g}] = [\mathfrak{h}, \mathfrak{k} \oplus \mathfrak{p}] = [\mathfrak{h}, \mathfrak{k}] + [\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{h}$. Hence \mathfrak{h} is an ideal of \mathfrak{g} .

Finally, we show that $\sigma(H) = H$ for every $\sigma \in \text{Aut}(G)$. Since $\sigma(K)$ is again a maximal compact subgroup of G , there exists $g \in G$ such that $c_g(K) = \sigma(K)$, where c_g is the inner automorphism of G induced by g . Since the automorphism $\tau := c_{g^{-1}} \circ \sigma$ satisfies $\tau(K) = K$, it follows from the definition of H that $\tau(H) = H$. Thus $\sigma(H) = c_g(\tau(H)) = c_g(H) = H$. This proves that H is a characteristic subgroup of G .

Step 2. We now prove the lemma for the case that G is semisimple without nontrivial connected compact characteristic subgroups. In this case, $G/Z(G)$ is again semisimple without nontrivial connected compact characteristic subgroups. Denote $n' = \text{ndim}(G/Z(G))$. Then $n = n' + \text{rank}(Z(G))$, $h = \text{rank}(Z(G))$. It follows that $n - h = n'$. By Step 1, we have

$$\dim G = \dim G/Z(G) \leq \frac{n'(n'+1)}{2} = \frac{(n-h)(n-h+1)}{2}.$$

This proves the lemma for such G .

Step 3. For the general case, let Q be the radical of G , $K \subseteq G/Q$ be the maximal connected compact characteristic subgroup of G/Q , and $R \subseteq G$ be the preimage of K under the quotient homomorphism $G \rightarrow G/Q$. Then R is a connected characteristic closed subgroup of G , and $G/R \cong (G/Q)/K$ is semisimple without nontrivial connected compact characteristic subgroups. Denote $n_1 = \text{ndim}(R)$, $n_2 = \text{ndim}(G/R)$, $h_2 = \text{hdim}(G/R)$. By Steps 1 and 2, there is a connected compact characteristic subgroup H of R such that

$$\dim R/H \leq \frac{n_1(n_1+1)}{2},$$

and we have

$$\dim G/R \leq \frac{(n_2 - h_2)(n_2 - h_2 + 1)}{2}.$$

The group H is also characteristic in G , and we have $n = n_1 + n_2$, $h = h_2$. It follows that

$$\begin{aligned} \dim G/H = \dim R/H + \dim G/R &\leq \frac{n_1(n_1 + 1)}{2} + \frac{(n_2 - h_2)(n_2 - h_2 + 1)}{2} \\ &\leq \frac{(n - h)(n - h + 1)}{2}. \end{aligned}$$

This completes the proof. \square

We now prove the first main result of this section.

Theorem 3.9. *Suppose G is a connected locally compact group with $\text{BM}(G) = r$. Then there is a connected compact normal subgroup H of G such that G/H is a Lie group of dimension at most $r(r + 1)/2$.*

Proof. This is a combination of Fact 3.6 and Lemma 3.8. \square

The following facts [12] come from the classification of Lie algebra of dimensions at most 2.

Fact 3.10. *Suppose G is a simply connected Lie group. We have the following:*

- (i) *if $\dim(G) = 1$, then $G = \mathbb{R}$;*
- (ii) *if $\dim(G) = 2$, then G is either \mathbb{R}^2 or the affine group of the line $\text{Aff}(1)$.*

Lemma 3.11. *Let G be a connected locally compact group with noncompact dimension n and helix dimension h . Then we have the following:*

- (i) *If $n - h = 1$, then there is a connected compact normal subgroup H of G such that G/H is \mathbb{R} .*
- (ii) *If $n - h = 2$, then there is a connected compact normal subgroup H of G such that G/H is either \mathbb{R}^2 , the affine group of the line, a semidirect product $\mathbb{R}^2 \rtimes \mathbb{T}$, or a cover of $\text{PSL}_2(\mathbb{R})$.*

Proof. To prove (i), we need only to notice that by Lemma 3.8, there is a connected compact normal subgroup H of G such that G/H is a connected Lie group with $\dim G/H \leq 1$. Since $n > 0$, G/H is noncompact, hence is isomorphic to \mathbb{R} .

We now prove (ii). In view of Lemma 3.8, Facts 3.1(ii) and 3.5(i), and quotienting out the maximal connected compact normal subgroup of G if necessary, we may assume that G is a connected Lie group without nontrivial connected compact normal subgroups such that $\dim G \leq 3$ and $n - h = 2$. Note that $\dim G \geq n \geq 2$. So $\dim G = 2$ or 3. If $\dim G = 2$, then $n = 2$, and it follows from the classification of 2-dimensional Lie algebras (Fact 3.10(ii)) that G is either \mathbb{R}^2 or the affine group

of the line. Assume that $\dim G = 3$. Then G is either semisimple or solvable. If G is semisimple, since it is noncompact, it must be a cover of $\mathrm{PSL}_2(\mathbb{R})$. Assume that G is solvable. Then $h = 0$, thus $n = 2$. It follows that G has a 1-dimensional maximal compact subgroup, say K , which must be isomorphic to \mathbb{T} . As in the proof of Lemma 3.8, we take an Ad_K -invariant subspace \mathfrak{p} of \mathfrak{g} complementary to \mathfrak{k} , and consider the subrepresentation $\rho : K \rightarrow \mathrm{GL}(\mathfrak{p})$ of Ad_K . Let us identify K with $\mathbb{T} = \{e^{i\theta} : \theta \in [0, 2\pi)\}$. Since $\dim \mathfrak{p} = 2$, there exist an integer k and a basis $\{X, Y\}$ of \mathfrak{p} such that the matrix of $\rho(e^{i\theta})$ is

$$\begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix}.$$

It follows that there is a nonzero vector $Z \in \mathfrak{k}$ such that $[Z, X] = kY$, $[Z, Y] = -kX$. If $k = 0$, then \mathfrak{k} is an ideal of \mathfrak{g} , which implies that K is normal in G , contradicting to the assumption that G has no nontrivial connected compact normal subgroups. So $k \neq 0$. It follows that $\mathfrak{p} \subset [\mathfrak{g}, \mathfrak{g}]$. Since \mathfrak{g} is solvable, we must have $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{p}$. Suppose $[X, Y] = aX + bY$, where $a, b \in \mathbb{R}$. By the Jacobi identity, we have

$$\begin{aligned} 0 &= [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] \\ &= [X, kX] + [Y, kY] + [Z, aX + bY] = akY - bkX. \end{aligned}$$

So $a = b = 0$. It follows that \mathfrak{p} is an abelian ideal of \mathfrak{g} . Thus $\mathfrak{g} = \mathfrak{p} \rtimes \mathfrak{k}$. This, together with the fact that K is maximal compact in G , implies that G is isomorphic to a semidirect product $\mathbb{R}^2 \rtimes \mathbb{T}$. \square

The next theorem is the second main result of the section.

Theorem 3.12. *Let G be a connected noncompact locally compact group, and let A, B be compact subsets of G . Then we have the following:*

- (i) *If $\mathrm{BM}(A, B) < 2$, then there is a continuous surjective group homomorphism with compact connected kernel mapping G to \mathbb{R} .*
- (ii) *If $\mathrm{BM}(A, B) < 3$, then there is a continuous surjective group homomorphism with compact connected kernel mapping G to either \mathbb{R} , \mathbb{R}^2 , $\mathbb{R}^2 \rtimes \mathbb{T}$, the affine group of the line, or a cover of $\mathrm{PSL}_2(\mathbb{R})$.*

If the nonabelian Brunn–Minkowski conjecture holds, we can strengthen the last case of (ii) into a finite cover of $\mathrm{PSL}_2(\mathbb{R})$.

Proof. Statements (i) and (ii) follow from Fact 3.6 and Lemma 3.11. If the nonabelian Brunn–Minkowski conjecture holds and the last case of (ii) is an infinite cover of $\mathrm{PSL}_2(\mathbb{R})$ (which must be $\widetilde{\mathrm{SL}_2(\mathbb{R})}$), then $\mathrm{BM}(A, B) \geq \mathrm{BM}(G) = \mathrm{ndim}(G) = \mathrm{ndim}(\widetilde{\mathrm{SL}_2(\mathbb{R})}) = 3$, a contradiction. \square

4. NONUNIMODULAR RUZSA DISTANCE AND OPEN APPROXIMATE GROUPS

In this section, we introduce a generalization of Ruzsa's distance in nonunimodular groups, which will be used in the next section. Throughout the section, G is a (not necessarily connected) locally compact group, μ_G is a left Haar measure, and ν_G is a right Haar measure on G . We will also assume that $\nu_G = \mu_G^{-1}$, but this assumption will not be needed until after Lemma 4.2. Suppose A, B are compact subsets of G of positive measures. Let

$$(4) \quad d(A, B) = \log \frac{\nu_G(AB^{-1})\mu_G(AB^{-1})}{\nu_G(A)\mu_G(B^{-1})}.$$

It is easy to see that $d(A, A) = 0$ if and only if A is a right translation of a subgroup of G . Thus for most of the sets A , we will have $d(A, A) > 0$. On the other hand, the distance still behaves like a metric in many aspects:

Lemma 4.1. *Let d be as in (4), and A, B, C are compact subsets of G with positive measures. Then we have*

- (i) (Nonnegativity) $d(A, B) \geq 0$;
- (ii) (Symmetric) $d(A, B) = d(B, A)$;
- (iii) (Left translation invariance) $d(A, B) = d(aA, bB)$ for every $a, b \in G$;
- (iv) (Triangle inequality) $d(A, B) \leq d(A, C) + d(C, B)$.

Note that when G is unimodular, by ignoring a constant factor, (4) becomes

$$d_R(A, B) = \log \frac{\mu_G(AB^{-1})}{\sqrt{\mu_G(A)\mu_G(B^{-1})}},$$

which is known as Ruzsa's distance. Lemma 4.1 for Ruzsa's distance was first established by Ruzsa [21] for discrete groups, and extended by Tao [24] to unimodular groups. Our proof below is similar as the one in [21, 24].

Proof of Lemma 4.1. Statement (i) is immediate from the definition. Statement (ii) is also clear in the special case where $\nu_G = \mu_G^{-1}$; the more general case follows from the uniqueness of Haar measure. Next, observe that

$$\begin{aligned} d(aA, bB) &= \log \frac{\nu_G(aAB^{-1})\mu_G(AB^{-1}b^{-1})}{\nu_G(aA)\mu_G(B^{-1}b^{-1})} \\ &= \log \frac{\Delta_G(a)^{-1}\nu_G(AB^{-1})\Delta_G(b)^{-1}\mu_G(AB^{-1})}{\Delta_G(a)^{-1}\nu_G(A)\Delta_G(b)^{-1}\mu_G(B^{-1})} = d(A, B), \end{aligned}$$

which proves (iii). Finally, to show the triangle inequality, it suffices to show that

$$(5) \quad \nu_G(AB^{-1})\mu_G(AB^{-1})\mu_G(C)\nu_G(C) \leq \nu_G(AC^{-1})\mu_G(AC^{-1})\nu_G(CB^{-1})\mu_G(CB^{-1}).$$

Using the right translation invariance of ν_G we get

$$\begin{aligned}\nu_G(AC^{-1})\nu_G(CB^{-1}) &= \iint_{G \times G} \mathbb{1}_{AC^{-1}}(x)\mathbb{1}_{CB^{-1}}(y) \, d\nu_G(x) \, d\nu_G(y) \\ &= \int_G \left(\int_G \mathbb{1}_{AC^{-1}}(zy^{-1})\mathbb{1}_{CB^{-1}}(y) \, d\nu_G(y) \right) \, d\nu_G(z).\end{aligned}$$

Note that when z is in AB^{-1} , there are $a \in A$ and $b \in B$ such that $z = ab^{-1}$. When $y \in Cb^{-1}$, clearly $\mathbb{1}_{AC^{-1}}(zy^{-1})\mathbb{1}_{CB^{-1}}(y) = 1$. As $\nu_G(Cb^{-1}) = \nu_G(C)$, we have

$$\nu_G(AC^{-1})\nu_G(CB^{-1}) \geq \int_G \mathbb{1}_{AB^{-1}}(z)\nu_G(C) \, d\nu_G(z) = \nu_G(AB^{-1})\nu_G(C).$$

Likewise, $\mu_G(AC^{-1})\mu_G(CB^{-1}) \geq \mu_G(AB^{-1})\mu_G(C)$, and this proves (5). \square

We will need the following nonunimodular version of Ruzsa's covering lemma:

Lemma 4.2. *Suppose A, B are compact subsets of G . Then we have the following:*

- (i) *If $\nu_G(AB) \leq K\nu_G(A)$, then there exists a finite subset $\Omega \subseteq B$ of cardinality at most K such that $B \subseteq A^{-1}A\Omega$.*
- (ii) *If $\mu_G(AB) \leq K\mu_G(B)$, then there exists a finite subset $\Omega \subseteq A$ of cardinality at most K such that $A \subseteq \Omega BB^{-1}$.*

Proof. We will only prove (i), as the proof of (ii) is the same. Let Ω be a maximal subset of B such that right translations of A by elements in Ω are pairwise disjoint. Thus by the right-translation invariance of ν_G , $|\Omega| \leq K$. By the maximality of Ω , for every $y \in B$, there is $y_0 \in \Omega$ such that $Ay \cap Ay_0 \neq \emptyset$. This implies $B \subseteq A^{-1}A\Omega$. \square

Given two compactly supported functions f and g on G , recall that the convolution $f * g$ is defined as

$$f * g(x) = \int_G f(y)g(y^{-1}x) \, d\mu_G(y).$$

When G is nonunimodular, in general we do not have $f * g(x) = \int_G f(xy^{-1})g(y) \, d\mu_G(y)$. Using the fact that $\nu_G = \mu_G^{-1}$, one can easily show that

$$f * g(x) = \int_G f(xy^{-1})g(y) \, d\nu_G(y).$$

We define the (right) multiplicative energy $E(A, B) = \int (\mathbb{1}_A * \mathbb{1}_B)^2 \, d\nu_G$. The following lemma gives us the relation between $E(A, A^{-1})$ and $E(A^{-1}, A)$.

Lemma 4.3. *Let A be a compact subset of G . Then*

$$E(A^{-1}, A) \geq \max_{a \in A} \frac{1}{\Delta(a)} E(A, A^{-1}).$$

Proof. We use $\widetilde{f}(x)$ to denote $f(x^{-1})$. Then from the definition we have

$$E(A^{-1}, A) = \int_G (\mathbb{1}_{A^{-1}} * \mathbb{1}_A(x))^2 d\nu_G(x) = \widetilde{\mathbb{1}_{A^{-1}} * \mathbb{1}_A} * \mathbb{1}_{A^{-1}} * \mathbb{1}_A(1).$$

Using the fact that $\widetilde{f * g} = \widetilde{g} * \widetilde{f}$, the right hand side of the above equality can be rewritten as

$$\widetilde{\mathbb{1}_A} * \widetilde{\mathbb{1}_{A^{-1}}} * \mathbb{1}_{A^{-1}} * \mathbb{1}_A(1),$$

which is the same as

$$\mathbb{1}_{A^{-1}} * \mathbb{1}_A * \mathbb{1}_{A^{-1}} * \mathbb{1}_A(1).$$

We now use the definition of the convolution, and the above quantity is equal to

$$\int_G \mathbb{1}_{A^{-1}}(y^{-1})(\mathbb{1}_A * \mathbb{1}_{A^{-1}} * \mathbb{1}_A(y)) d\nu_G(y).$$

By Fact 2.2 (iv), the above quantity can be rewritten as

$$\int_G \mathbb{1}_{A^{-1}}(y^{-1}) \mathbb{1}_A * \mathbb{1}_{A^{-1}} * \mathbb{1}_A(y) \frac{1}{\Delta(y)} d\mu_G(y).$$

Hence we have

$$E(A^{-1}, A) \geq \max_{a \in A} \frac{1}{\Delta(a)} \mathbb{1}_A * \mathbb{1}_{A^{-1}} * \mathbb{1}_A * \mathbb{1}_{A^{-1}}(1).$$

On the other hand, by a similar argument, we have

$$E(A, A^{-1}) = \widetilde{\mathbb{1}_A * \mathbb{1}_{A^{-1}}} * \mathbb{1}_A * \mathbb{1}_{A^{-1}}(1) = \mathbb{1}_A * \mathbb{1}_{A^{-1}} * \mathbb{1}_A * \mathbb{1}_{A^{-1}}(1),$$

and the lemma follows. \square

As a corollary of Lemma 4.3, one can see that when G is *unimodular*, we have

$$E(A, A^{-1}) = E(A^{-1}, A).$$

Definition 4.4 (Approximate groups). An open and precompact set $A \subseteq G$ is a *K-approximate group*, if $A = A^{-1}$, $\text{id}_G \in A$, and $A^2 \subseteq \Omega A$ for some finite set Ω of cardinality at most K .

We define the (right-translation) approximate measure stabilizer

$$\text{Stab}_{\nu_G}^\varepsilon(A) = \{g \in G : \nu_G(A \setminus Ag) < \varepsilon\},$$

where $\varepsilon < \nu_G(A)$. We have the following simple observation, and the proof is straightforward.

Fact 4.5. *If A is compact, then $S = \text{Stab}_G^\varepsilon(A)$ contains 1_G and is open, precompact, and equal to S^{-1} .*

The following proposition is a nonunimodular analog of [24, Proposition 4.5]. The proof is similar as the one in [24], with certain modifications for the nonunimodular groups.

Proposition 4.6. *Suppose A is a compact subset of G with $\nu_G(AA^{-1}) \leq K\mu_G(A)$ and $\max_{a \in A} \Delta(a) = 1$. Let*

$$S = \text{Stab}_{\nu_G}^{(2K-1)\nu_G(A)/2K}(A).$$

Then we have the following:

- (i) S is symmetric and $S \subseteq A^{-1}A$;
- (ii) $\nu_G(S) \geq \mu_G(A)/2K$;
- (iii) for all integers $n \geq 1$, $\nu_G(AS^nA^{-1}) \leq 2^n K^{2n+1} \mu_G(A)$.

Proof. (i) is straightforward by the way we construct S . Indeed, for every $s \in S$, $A \cap As \neq \emptyset$ and hence $S \subseteq A^{-1}A$. By the Cauchy–Schwarz inequality, we have

$$E(A^{-1}, A) \geq E(A, A^{-1}) \geq \frac{\nu_G(A)^2 \mu_G(A)^2}{\nu_G(AA^{-1})} \geq \frac{\nu_G(A)^2 \mu_G(A)}{K}.$$

Also by the choice of S , we have

$$\int_{G \setminus S} (\mathbb{1}_{A^{-1}} * \mathbb{1}_A(x))^2 d\nu_G(x) \leq \frac{\nu_G(A)}{2K} \int_G \mathbb{1}_{A^{-1}} * \mathbb{1}_A(x) d\nu_G(x) \leq \frac{\nu_G(A)^2 \mu_G(A)}{2K}.$$

Thus we have

$$\int_S (\mathbb{1}_{A^{-1}} * \mathbb{1}_A(x))^2 d\nu_G(x) \geq \frac{\nu_G(A)^2 \mu_G(A)}{2K}.$$

This gives us $\nu_G(S) \geq \mu_G(A)/2K$, which proves (ii).

Next we prove statement (iii). Note that we have

$$(6) \quad \int_{G^{n+1}} \mathbb{1}_{AS^nA^{-1}}(y_0 \cdots y_n) \prod_{i=0}^n (\mathbb{1}_{AA^{-1}}(y_i) d\nu_G(y_i))$$

$$(7) \quad = \int_{G^{n+1}} \mathbb{1}_{AS^nA^{-1}}(x) \mathbb{1}_{AA^{-1}}(xy_n^{-1} \cdots y_1^{-1}) \prod_{i=1}^n (\mathbb{1}_{AA^{-1}}(y_i) d\nu_G(y_i)) d\nu_G(x).$$

Fix an arbitrary $x \in AS^nA^{-1}$. Then we can fix $a, b_n \in A$ and $s_1, \dots, s_n \in S$ such that $x = as_1 \cdots s_n b_n^{-1}$. For $b_0, \dots, b_{n-1} \in G$, if we set $y_i = b_{i-1} s_i b_i^{-1}$ for $1 \leq i \leq n$, then $xy_n^{-1} \cdots y_1^{-1} = ab_0^{-1}$. Thus

$$\begin{aligned} & \int_{G^n} \mathbb{1}_{AA^{-1}}(xy_n^{-1} \cdots y_1^{-1}) \prod_{i=1}^n (\mathbb{1}_{AA^{-1}}(y_i) d\nu_G(y_i)) \\ &= \int_{G^n} \mathbb{1}_{AA^{-1}}(ab_0^{-1}) \prod_{i=1}^n (\mathbb{1}_{AA^{-1}}(b_{i-1} s_i b_i^{-1}) d\nu_G(b_{i-1})). \end{aligned}$$

The above integrand is 1 if $b_i \in A$ and $b_i s_{i+1} \in A$ for $0 \leq i \leq n-1$. Hence, the above integral is at least

$$\int_{G^n} \prod_{i=0}^{n-1} \mathbb{1}_A(b_i) \mathbb{1}_A(b_i s_{i+1}) d\nu_G(b_i) = \prod_{i=0}^{n-1} \nu_G(A \cap A s_{i+1}^{-1}) \geq \left(\frac{\nu_G(A)}{2K} \right)^n$$

as s_i are from the approximate measure stabilizer S . Thus, (7) is at least

$$\nu_G(AS^n A^{-1})(\nu_G(A)/2K)^n.$$

On the other hand, (6) is bounded from above by

$$\nu_G(AA^{-1})^{n+1} \leq K^{n+1} \mu_G(A)^{n+1}.$$

Note that as we have

$$\nu_G(A) = \int_G \mathbb{1}_A(x) \frac{1}{\Delta_G(x)} \mu_G(A) \geq \mu_G(A),$$

so $\nu_G(AA^{-1})^{n+1} \leq K^{n+1} \mu_G(A) \nu_G(A)^n$. This proves the proposition. \square

The next theorem says that sets with small Ruzsa's distance are commensurable to approximate groups. This is the nonunimodular counterpart of [24, Theorem 4.6].

Theorem 4.7. *Suppose A, B are compact subsets of G , and $d(A, B^{-1}) \leq \log K$. Then there is $H \subseteq G$ satisfying:*

- (i) (group-like) H is a $64K^{12}$ -approximate group;
- (ii) (smallness) $\mu_G(H) = \nu_G(H) \leq 4K^5(\nu_G(A)\mu_G(B))^{\frac{1}{2}}$;
- (iii) (largeness) $A \subseteq \Omega H$ and $B \subseteq H\Omega$ where $|\Omega| \leq 33K^{12}$.

Proof. Choose $g_1 \in A$, and $g_2 \in B$ such that $\Delta_G(g_1) = \max_{a \in A} \Delta_G(a)$. Replacing A with $g_1^{-1}A$ and B with Bg_2^{-1} if necessary, we can arrange that $\text{id}_G \in A \cap B$ and $\max_{a \in A} \Delta(a) = 1$.

By Lemma 4.1,

$$d(A, A) \leq 2d(A, B^{-1}) \leq 2 \log K,$$

that is $\nu_G(AA^{-1}) \leq K\mu_G(A)$ since we chose $\nu_G = \mu_G^{-1}$. Let

$$S = \text{Stab}_{\nu_G}^{(2K-1)\nu_G(A)/2K}(A).$$

Note that $S^n \subseteq AS^n A^{-1}$. By Proposition 4.6, we have

$$\nu_G(S^5) \leq \nu_G(AS^5 A^{-1}) \leq 32K^{11} \mu_G(A) \leq 64K^{12} \nu_G(S).$$

As S is symmetric, by Lemma 4.2, S^2 is a $64K^{12}$ -approximate group. Let $H = S^2$. Also by Proposition 4.6 again we have

$$\nu_G(H) \leq \nu_G(AHA^{-1}) \leq 4K^5 \nu_G(A).$$

By Proposition 4.6 again and the fact that $\text{id}_G \in A$,

$$\nu_G(HA^{-1}) \leq \nu_G(AHA^{-1}) \leq 4K^5 \mu_G(A) \leq 8K^6 \nu_G(S) \leq 8K^6 \nu_G(H).$$

Hence by Lemma 4.2, there is $\Omega_1 \subseteq A$ of cardinality at most $8K^6$ such that $A \subseteq \Omega_1 H$.

By Lemma 4.1 again, $d(B^{-1}, H) \leq d(A, B^{-1}) + d(A, H)$, and

$$d(A, H) = \log \frac{\nu_G(AH) \mu_G(AH)}{\nu_G(A) \mu_G(H)} \leq \log \frac{4K^5 \nu_G(A) 8K^6 \mu_G(H)}{\nu_G(A) \mu_G(H)} = \log 32K^{11}.$$

This implies that

$$\nu_G(HB) \leq 32K^{12} \nu_G(H) \frac{\mu_G(B)}{\mu_G(HB)} \leq 32K^{12} \nu_G(H).$$

Thus by Lemma 4.2 again, there is $\Omega_2 \subseteq B$ of cardinality at most $32K^{12}$ such that $B \subseteq H\Omega_2$. Set $\Omega = g_1^{-1}\Omega_1 \cup \Omega_2 g_2^{-1}$. Note that by Kemperman's inequality, we may assume $K \geq 4$, and hence $|\Omega| \leq 33K^{12}$ which finishes the proof. \square

5. QUOTIENT DOMINATIONS

In this section, we assume there is a short exact sequence

$$1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1,$$

where H is a compact subgroup of G . We first consider the case when sets (A, B) has minimal measure expansion on G . In this case, we will have a quotient domination result.

Theorem 5.1 (Quotient dominations). *Let G be a connected unimodular noncompact group, and $\text{BM}(G) = n$. Let A, B be compact subsets in G such that*

$$\mu_G(AB)^{\frac{1}{n}} = \mu_G(A)^{\frac{1}{n}} + \mu_G(B)^{\frac{1}{n}}.$$

Suppose $\pi : G \rightarrow G/H$ is the quotient map. Then there are compact sets A', B' in G/H with

$$\mu_{G/H}(A'B')^{\frac{1}{n}} = \mu_{G/H}(A')^{\frac{1}{n}} + \mu_{G/H}(B')^{\frac{1}{n}},$$

and $A = \pi^{-1}(A')$, $B = \pi^{-1}(B')$.

Proof. Define the fiber length function $G/H \rightarrow \mathbb{R}$:

$$f_A(g) = \mu_H(g^{-1}A \cap H).$$

We similarly define fiber length functions for B and for AB . We use L_f^+ to denote the superlevel set of f , that is

$$L_f^+(t) = \{x \in G : f(x) \geq t\}.$$

From the fact that μ_G can be pushed forward to a Haar measure on G/H , one can see that $\text{BM}(G/H) \geq n$. It is clear that $\text{BM}(H) = 0$. Hence,

$$(8) \quad \mu_{G/H}^{1/n} (L_{f_{AB}}^+ (\max\{t_1, t_2\})) \geq \mu_{G/H}^{1/n} (L_{f_A}^+ (t_1)) + \mu_{G/H}^{1/n} (L_{f_B}^+ (t_2)).$$

Given a compact set Ω in G , we define

$$E_\Omega(t) = \mu_{G/H}^{1/n} (L_{f_\Omega}^+ (t)).$$

Thus by (8), we have $E_{AB}(\max\{t_1, t_2\}) \geq E_A(t_1) + E_B(t_2)$ for all $t_1, t_2 \in \mathbb{R}$. It then follows that

$$(9) \quad \mu_{\mathbb{R}}(L_{E_{AB}}^+ (s_1 + s_2)) \geq \max\{\mu_{\mathbb{R}}(L_{E_A}^+ (s_1)), \mu_{\mathbb{R}}(L_{E_B}^+ (s_2))\}.$$

Now we consider $\mu_G(AB)$. We have

$$(10) \quad \mu_G^{1/n}(AB) = \left(\int_{\mathbb{R} > 0} E_{AB}^n(s) ds \right)^{1/n} = \left(\int_{\mathbb{R} > 0} n s^{n-1} \mu_{\mathbb{R}}(L_{E_{AB}}^+ (s)) ds \right)^{1/n}$$

Note that $\sup_t E_A(t) = \mu_{G/H}(\pi A)$ and $\sup_t E_B(t) = \mu_{G/H}(\pi B)$. Then $n^{-1/n} \mu_G^{1/n}(AB)$ is at least

$$\left((\mu_{G/H}(\pi A) + \mu_{G/H}(\pi B))^n \int_0^1 s^{n-1} \mu_{\mathbb{R}}(L_{E_{AB}}^+ (\mu_{G/H}(\pi A) + \mu_{G/H}(\pi B))s) ds \right)^{\frac{1}{n}}.$$

Using (9), the above quantity is at least

$$\left((\mu_{G/H}(\pi A) + \mu_{G/H}(\pi B))^n \cdot \max \left\{ \int_0^1 s^{n-1} \mu_{\mathbb{R}}(L_{E_A}^+ (\mu_{G/H}(\pi A)s)) ds, \int_0^1 s^{n-1} \mu_{\mathbb{R}}(L_{E_B}^+ (\mu_{G/H}(\pi B)s)) ds \right\} \right)^{1/n}$$

Now by Hölder's inequality, the above quantity is at least

$$\begin{aligned} & \left(\mu_{G/H}(\pi A)^n \int_0^1 s^{n-1} \mu_{\mathbb{R}}(L_{E_A}^+ (\mu_{G/H}(\pi A)s)) ds \right)^{1/n} \\ & + \left(\mu_{G/H}(\pi B)^n \int_0^1 s^{n-1} \mu_{\mathbb{R}}(L_{E_B}^+ (\mu_{G/H}(\pi B)s)) ds \right)^{1/n}. \end{aligned}$$

It is easy to check the above quantity is equal to

$$n^{-1/n} \mu_G^{1/n}(A) + n^{-1/n} \mu_G^{1/n}(B).$$

Since we have that $\mu_G(AB)^{1/n} = \mu_G(A)^{1/n} + \mu_G(B)^{1/n}$, equalities hold everywhere in the above inequalities. This in particular implies that

$$\mu_{G/H}(\pi A \pi B)^{\frac{1}{n}} = \mu_{G/H}(\pi A)^{\frac{1}{n}} + \mu_{G/H}(\pi B)^{\frac{1}{n}},$$

and $\mu_G(A) = \mu_{G/H}(\pi A)$, $\mu_G(B) = \mu_{G/H}(\pi B)$. By the compactness assumption of A and B , we have $A = \pi^{-1}\pi A$ and $B = \pi^{-1}\pi B$. \square

We introduce the notion of discrepancy, which is standard in the literature.

Definition 5.2 (discrepancy). Let G be a unimodular group, and μ_G a Haar measure on G . Given compact sets $A, B \subseteq G$, the *discrepancy* of A and B , denoted by $\mathfrak{d}_G(A, B)$, is $\mu_G(AB) - \mu_G(A) - \mu_G(B)$.

When $\mathfrak{d}_G(A, B)$ is small, we use the following theorem [14, Theorem 6.5] by the middle two authors.

Theorem 5.3 (Almost quotient dominations). *Let G be a connected noncompact unimodular group, and A, B be compact subsets in G such that*

$$\mathfrak{d}_G(A, B) < \min\{\mu_G(A), \mu_G(B)\}.$$

Then there are compact sets A', B' in G/H with

$$\mathfrak{d}_{G/H}(A', B') < 7\mathfrak{d}_G(A, B),$$

and $\mu_G(A \Delta \pi^{-1}(A')) \leq 3\mathfrak{d}_G(A, B)$, $\mu_G(B \Delta \pi^{-1}(B')) \leq 3\mathfrak{d}_G(A, B)$.

Let $\pi : G \rightarrow G/H$ be the quotient map. For an open approximate group A on G , one can easily see that πA is also an open approximate group on G/H . The next proposition show that small expansion sets have a similar behaviour.

Proposition 5.4. *Let G be a connected noncompact unimodular group, and A a compact subset of G with positive measure. Suppose $\mu_G(A^2) \leq K\mu_G(A)$, and $\pi : G \rightarrow G/H$ is the quotient map. Then $\mu_{G/H}(\pi A^2) \leq 64K^6\mu_{G/H}(\pi A)$.*

Proof. Let $f_A : G/H \rightarrow \mathbb{R}$ be the fiber length function that $f_A(g) = \mu_H(g^{-1}A \cap H)$. Set $\alpha = \sup_g f_A(g)$, clearly $\alpha \leq 1$. Set $N = 2K$. Define

$$\pi A_1 = \left\{ g \in G/H : f_A(g) \geq \frac{\alpha}{N} \right\}, \quad \text{and} \quad \pi A_2 = \pi A \setminus \pi A_1.$$

Note that $K\mu_G(A) \geq \mu_G(A^2) \geq \alpha\mu_{G/H}(\pi A) \geq \mu_G(A) + (1 - 1/N)\alpha\mu_{G/H}(\pi A_2)$. Hence,

$$\begin{aligned} \mu_{G/H}(\pi A_2) &\leq \frac{K-1}{\alpha} \frac{N}{N-1} \mu_G(A) \\ &\leq \frac{K-1}{\alpha} \frac{N}{N-1} \left(\alpha\mu_{G/H}(\pi A_1) + \frac{\alpha}{N} \mu_{G/H}(\pi A_2) \right). \end{aligned}$$

By the choice of N , we have

$$\mu_{G/H}(\pi A_2) \leq 2(K-1)\mu_{G/H}(\pi A_1).$$

On the other hand, we have

$$\begin{aligned}\mu_{G/H}(\pi A_1 \pi A_1) &\leq \frac{N}{\alpha} \mu_G(A^2) \\ &\leq \frac{KN}{\alpha} \left(\alpha \mu_{G/H}(\pi A_1) + \frac{\alpha}{N} \mu_{G/H}(\pi A_2) \right) \leq 2K(2K-1) \mu_{G/H}(\pi A_1).\end{aligned}$$

For $\mu_{G/H}(\pi A \pi A_1)$ and $\mu_{G/H}(\pi A_1 \pi A)$, we use similar ideas to get

$$\begin{aligned}\max\{\mu_{G/H}(\pi A \pi A_1), \mu_{G/H}(\pi A_1 \pi A)\} &\leq \frac{N}{\alpha} \mu_G(A^2) \\ &\leq 2K(2K-1) \mu_{G/H}(\pi A_1) \\ &\leq 2K(2K-1) (\mu_{G/H}(\pi A_1) \mu_{G/H}(\pi A))^{\frac{1}{2}}.\end{aligned}$$

Let d be the Ruzsa distance defined in (4). Hence $d(\pi A_1, (\pi A_1)^{-1}) \leq 2 \log 2K(2K-1)$, and $\max\{d(\pi A_1, (\pi A)^{-1}), d(\pi A, (\pi A_1)^{-1})\} \leq 2 \log 2K(2K-1)$. Using Lemma 4.1,

$$\begin{aligned}d(\pi A, (\pi A)^{-1}) &\leq d(\pi A, (\pi A_1)^{-1}) + d((\pi A_1)^{-1}, \pi A_1) + d(\pi A_1, (\pi A)^{-1}) \\ &\leq 2 \log 8K^3(2K-1)^3.\end{aligned}$$

This implies $\mu_{G/H}(\pi A^2) \leq 64K^6 \mu_{G/H}(\pi A)$. \square

It might be of interest to improve the constant $64K^6$ in the above lemma.

6. MINIMAL EXPANSIONS IN NONUNIMODULAR GROUPS

In this section, we will study the sets with minimal expansions in nonunimodular groups. Suppose G is a nonunimodular group with $\text{BM}(G) = 2$, the Brunn–Minkowski inequality given in [15] asserts that for any compact sets A, B one has

$$\left(\frac{\nu_G(A)}{\nu_G(AB)} \right)^{\frac{1}{2}} + \left(\frac{\mu_G(B)}{\mu_G(AB)} \right)^{\frac{1}{2}} \leq 1.$$

Our main result in the section is the following theorem, which shows that the equality in the above inequality cannot hold under certain conditions.

Proposition 6.1. *Let G be a nonunimodular connected locally compact group, and A is a compact subset of G with positive measure. Then*

$$\left(\frac{\nu_G(A)}{\nu_G(A^2)} \right)^{\frac{1}{2}} + \left(\frac{\mu_G(A)}{\mu_G(A^2)} \right)^{\frac{1}{2}} < 1.$$

Proof. Note that G is nonunimodular. Let $H = \ker \Delta_G$, then $G/H \cong \mathbb{R}^{>0}$. As G is nonunimodular, H is noncompact. Note also that H is unimodular. Let $\mu_{\mathbb{R}}$ be a

Haar measure on $\mathbb{R}^{>0}$ with respect to multiplication. Define the right and left fiber length functions $G/H \rightarrow \mathbb{R}$:

$$r_A(g) = \mu_H(Ag^{-1} \cap H), \quad \ell_A(g) = \mu_H(g^{-1}A \cap H).$$

In particular, we have

$$(11) \quad \ell_A(g) = \Delta_G(g)^{-1}r_A(g).$$

We similarly define the right and left fiber length function for A^2 . Note that for compact sets $X_1, X_2 \subseteq H$, and $g_1, g_2 \in G$, $X_1g_1g_2X_2 \subseteq g_1g_2H = Hg_1g_2$ as H is normal. By Kemperman's inequality on H and Fact 2.3, we have

$$(12) \quad \begin{aligned} \mu_H((g_1g_2)^{-1}X_1g_1g_2X_2) &\geq \mu_H((g_1g_2)^{-1}X_1g_1g_2) + \mu_H(X_2) \\ &= \Delta_G((g_1g_2)^{-1})\mu_H(X_1) + \mu_H(X_2), \end{aligned}$$

and similarly,

$$(13) \quad \begin{aligned} \mu_H(X_1g_1g_2X_2(g_1g_2)^{-1}) &\geq \mu_H(X_1) + \mu_H(g_1g_2X_2(g_1g_2)^{-1}) \\ &= \mu_H(X_1) + \Delta_G(g_1g_2)\mu_H(X_2). \end{aligned}$$

Now we consider $\nu_G(A^2)$. By the quotient integral formula, we have

$$\nu_G(A^2) = \int_{G/H} r_{A^2}(g) \, d\mu_{\mathbb{R}}(g).$$

Note that $(Hg)^2 = Hg^2$ as H is normal, we have the natural lower bound of the right hand side

$$\int_{G/H} r_{A^2}(g^2) \mathbb{1}_{\text{supp}(r_A)}(g) \, d\mu_{\mathbb{R}}(g^2),$$

and by (13) this is at least

$$2 \int_{G/H} r_A(g) + \Delta_G(g^2)\ell_A(g) \, d\mu_{\mathbb{R}}(g) = 2\nu_G(A) + 2 \int_{G/H} \Delta_G(g^2)\ell_A(g) \, d\mu_{\mathbb{R}}(g).$$

Similarly, using the quotient integral formula we obtain

$$\mu_G(A^2) = \int_{G/H} \ell_{A^2}(g) \, d\mu_{\mathbb{R}}(g) \geq \int_{G/H} \ell_{A^2}(g^2) \mathbb{1}_{\text{supp}(\ell_A)}(g) \, d\mu_{\mathbb{R}}(g^2).$$

Now by (12) the right hand side at least

$$2 \int_{G/H} \Delta_G(g^2)^{-1}r_A(g) + \ell_A(g) \, d\mu_{\mathbb{R}}(g) = 2\mu_G(A) + 2 \int_{G/H} \Delta_G(g^2)^{-1}r_A(g) \, d\mu_{\mathbb{R}}(g).$$

By applying the above lower bounds on $\nu(A^2)$ and $\mu(A^2)$, we have

$$\left(\frac{\nu_G(A)}{\nu_G(A^2)}\right)^{\frac{1}{2}} + \left(\frac{\mu_G(A)}{\mu_G(A^2)}\right)^{\frac{1}{2}} \leq \left(\frac{1}{2 + \frac{2 \int \Delta_G(g^2) \ell_A(g) d\mu_{\mathbb{R}}}{\nu_G(A)}}\right)^{\frac{1}{2}} + \left(\frac{1}{2 + \frac{2 \int \Delta_G(g^2)^{-1} r_A(g) d\mu_{\mathbb{R}}}{\mu_G(A)}}\right)^{\frac{1}{2}}.$$

Using the fact that $\nu_G(A) = \int_{G/H} r_A(g) d\mu_{\mathbb{R}}$ and $\mu_G(A) = \int_{G/H} \ell_A(g) d\mu_{\mathbb{R}}$, together with (11) and Hölder's inequality, we have that the right hand side of the above quantity is at most

$$\frac{\int_{G/H} \Delta_G(g) \ell_A(g) d\mu_{\mathbb{R}}}{\int_{G/H} \Delta_G(g) \ell_A(g) d\mu_{\mathbb{R}} + \int_{G/H} \Delta_G(g)^2 \ell_A(g) d\mu_{\mathbb{R}}} + \frac{\int_{G/H} \ell_A(g) d\mu_{\mathbb{R}}}{\int_{G/H} \ell_A(g) d\mu_{\mathbb{R}} + \int_{G/H} \Delta_G(g)^{-1} \ell_A(g) d\mu_{\mathbb{R}}}$$

Note that the above quantity can be written as

$$(14) \quad \begin{aligned} & \left(\int_{G/H} \ell_A(g) d\mu_{\mathbb{R}} \int_{G/H} \Delta_G(g) \ell_A(g) d\mu_{\mathbb{R}} + \int_{G/H} \Delta_G(g)^{-1} \ell_A(g) d\mu_{\mathbb{R}} \int_{G/H} \Delta_G(g) \ell_A(g) d\mu_{\mathbb{R}} \right. \\ & \left. + \int_{G/H} \ell_A(g) d\mu_{\mathbb{R}} \int_{G/H} \Delta_G(g)^2 \ell_A(g) d\mu_{\mathbb{R}} + \int_{G/H} \ell_A(g) d\mu_{\mathbb{R}} \int_{G/H} \Delta_G(g) \ell_A(g) d\mu_{\mathbb{R}} \right) \\ & \cdot \left(\int_{G/H} \ell_A(g) d\mu_{\mathbb{R}} \int_{G/H} \Delta_G(g) \ell_A(g) d\mu_{\mathbb{R}} + \int_{G/H} \Delta_G(g)^{-1} \ell_A(g) d\mu_{\mathbb{R}} \int_{G/H} \Delta_G(g) \ell_A(g) d\mu_{\mathbb{R}} \right. \\ & \left. + \int_{G/H} \ell_A(g) d\mu_{\mathbb{R}} \int_{G/H} \Delta_G(g)^2 \ell_A(g) d\mu_{\mathbb{R}} + \int_{G/H} \Delta_G(g)^{-1} \ell_A(g) d\mu_{\mathbb{R}} \int_{G/H} \Delta_G(g)^2 \ell_A(g) d\mu_{\mathbb{R}} \right)^{-1} \end{aligned}$$

Using Hölder's inequality again, we have

$$(15) \quad \begin{aligned} & \int_{G/H} \Delta_G(g) \ell_A(g) d\mu_{\mathbb{R}} \int_{G/H} \ell_A(g) d\mu_{\mathbb{R}} \\ & \leq \left(\int_{G/H} \Delta_G(g)^2 \ell_A(g) d\mu_{\mathbb{R}} \right)^{\frac{2}{3}} \left(\int_{G/H} \Delta_G(g)^{-1} \ell_A(g) d\mu_{\mathbb{R}} \right)^{\frac{1}{3}} \\ & \quad \cdot \left(\int_{G/H} \Delta_G(g)^2 \ell_A(g) d\mu_{\mathbb{R}} \right)^{\frac{1}{3}} \left(\int_{G/H} \Delta_G(g)^{-1} \ell_A(g) d\mu_{\mathbb{R}} \right)^{\frac{2}{3}} \\ & = \int_{G/H} \Delta_G(g)^2 \ell_A(g) d\mu_{\mathbb{R}} \int_{G/H} \Delta_G(g)^{-1} \ell_A(g) d\mu_{\mathbb{R}}. \end{aligned}$$

By applying (15) to the second term of the second line of (14), we conclude that

$$(16) \quad \left(\frac{\nu_G(A)}{\nu_G(A^2)}\right)^{\frac{1}{2}} + \left(\frac{\mu_G(A)}{\mu_G(A^2)}\right)^{\frac{1}{2}} \leq 1.$$

When the equality holds in (16), equalities hold in all the inequalities in the above estimates. In particular, equality holds in (15) implies that $\Delta_G(g)$ is a constant for almost all $g \in \text{supp}(\ell_A)$. Since G is connected and nonunimodular, $\ker \Delta_G$ has measure 0, this would imply that A has measure 0, and hence contradicts the fact that A has positive measure. \square

7. PROOF OF THE MAIN THEOREMS

Our first theorem is the asymmetric version of Theorem 1.1:

Theorem 7.1. *Let G be a connected noncompact locally compact group, μ is a left Haar measure, and ν is a right Haar measure on G . Suppose A, B are compact subsets of G of positive measures, and*

$$(17) \quad d(A, B^{-1}) < \log K,$$

where d is the nonunimodular Ruzsa distance defined in (4). Then there is a continuous surjective group homomorphism $\chi : G \rightarrow L$ with compact kernel, where L is a connected Lie group of dimension at most $\lfloor \log K/2 \rfloor (1 + \lfloor \log K/2 \rfloor)/2$, and an open $64K^{12}$ -approximate group X of G' , and A can be covered by at most $33K^{12}$ left translations of $\chi^{-1}(X)$, B can be covered by at most $33K^{12}$ right translations of $\chi^{-1}(X)$. Moreover, if $A = B$ and G is unimodular, with $A' = \chi(A)$, we have $\mu_L((A')^2) \leq 64K^3 \mu_L(A')$.

Proof. As $d(A, B^{-1}) < \log K$, we have

$$\left(\frac{\nu_G(A)}{\nu_G(AB)} \right)^{\frac{2}{\log K}} + \left(\frac{\mu_G(B)}{\mu_G(AB)} \right)^{\frac{2}{\log K}} \geq \left(K \frac{\nu_G(A)\mu_G(B)}{\nu_G(AB)\mu_G(AB)} \right)^{\frac{1}{\log K}} \geq 1,$$

which implies that $\text{BM}(G) \leq \lfloor \log K/2 \rfloor$. Hence by Theorem 3.9, there is a continuous surjective group homomorphism $\chi : G \rightarrow L$ with compact kernel, and L is a connected Lie group satisfying

$$\dim(L) \leq \frac{\lfloor \log K/2 \rfloor (1 + \lfloor \log K/2 \rfloor)}{2}.$$

By Theorem 4.7, there is a $64K^{12}$ open approximate group H of G , such that $A \subseteq \Omega H$ and $B \subseteq H\Omega$ for some finite set Ω of cardinality at most $33K^{12}$. Let $X = \chi(H)$. It is easy to see that X is the desired $64K^{12}$ -approximate group. When $A = B$ the desired conclusion follows from Proposition 5.4. \square

Next, we prove the nearly minimal measure expansions result:

Proof of Theorem 1.3. Using Theorem 3.12, we know that G is unimodular, and moreover, there is a surjective continuous group homomorphism $\chi : G \rightarrow \mathbb{R}$ with compact kernel. We first consider the case when $\mu_G(AB) = \mu_G(A) + \mu_G(B)$. By

Theorem 5.1, there are compact sets $X, Y \subseteq \mathbb{R}$ with $\mu_{\mathbb{R}}(X + Y) = \mu_{\mathbb{R}}(X) + \mu_{\mathbb{R}}(Y)$, such that

$$\mu_G(A) = \mu_{\mathbb{R}}(X), \quad \mu_G(B) = \mu_{\mathbb{R}}(Y),$$

and $A \subseteq \chi^{-1}(X)$, $B \subseteq \chi^{-1}(Y)$. By Fact 2.5, there are compact intervals $I, J \subseteq \mathbb{R}$, such that $X = I$ and $Y = J$. Hence $A \subseteq \chi^{-1}(I)$ and $B \subseteq \chi^{-1}(J)$, this proves statement (ii).

Now we prove statement (i). By Theorem 5.3, there are compact sets $X, Y \subseteq \mathbb{R}$, with

$$\mu_G(A \triangle \chi^{-1}(X)) \leq 3\mathfrak{d}_G(A, B), \quad \mu_G(B \triangle \chi^{-1}(Y)) \leq 3\mathfrak{d}_G(A, B),$$

and

$$\mathfrak{d}_{\mathbb{R}}(X, Y) \leq 7\mathfrak{d}_G(A, B).$$

Assume that $\mathfrak{d}_G(A, B) < 1/20 \min\{\mu_G(A), \mu_G(B)\}$, then we get

$$\mathfrak{d}_{\mathbb{R}}(X, Y) < \frac{1}{2} \min\{\mu_G(A), \mu_G(B)\} \leq \min\{\mu_{\mathbb{R}}(X), \mu_{\mathbb{R}}(Y)\}.$$

Applying Fact 2.5, we find intervals $I'', J'' \subseteq \mathbb{R}$ with $X \subseteq I''$ and $Y \subseteq J''$ such that

$$\mu_{\mathbb{R}}(I'') \leq \mu_{\mathbb{R}}(X + Y) - \mu_{\mathbb{R}}(Y) \text{ and } \mu_{\mathbb{R}}(J'') \leq \mu_{\mathbb{R}}(X + Y) - \mu_{\mathbb{R}}(X).$$

In particular, we can choose subintervals $I' \subseteq I''$ and $J' \subseteq J''$ with $\mu_{\mathbb{R}}(I') = \mu_{\mathbb{R}}(X)$, $\mu_{\mathbb{R}}(J') = \mu_{\mathbb{R}}(Y)$, and

$$\mu_{\mathbb{R}}(I' \triangle X) \leq \mathfrak{d}_{\mathbb{R}}(X, Y) \quad \mu_{\mathbb{R}}(J' \triangle Y) \leq \mathfrak{d}_{\mathbb{R}}(X, Y).$$

For any element $g \in A \setminus \chi^{-1}(I')$, assume that the Euclidean distance between $\chi(g)$ and I' in \mathbb{R} is strictly greater than $50\mathfrak{d}_G(A, B)$. We then have

$$\mu_{\mathbb{R}}(\chi(g)\chi(B) \setminus I'\chi(B)) \geq 50\mathfrak{d}_G(A, B) - \mu_{\mathbb{R}}(J \setminus \chi(B)) \geq 40\mathfrak{d}_G(A, B).$$

This implies that $\mu_G(gB \setminus \chi^{-1}(I')\chi^{-1}(J')) \geq 30\mathfrak{d}_G(A, B)$. Therefore,

$$\begin{aligned} \mu_G(AB) &\geq \mu_G((\chi^{-1}(I') \cap A)(\chi^{-1}(J') \cap B)) + \mu_G(gB \setminus \chi^{-1}(I')\chi^{-1}(J')) \\ &\geq \mu_G(A) + \mu_G(B) - 20\mathfrak{d}_G(A, B) + 30\mathfrak{d}_G(A, B), \end{aligned}$$

and this is a contradiction. Thus there are intervals $I \supseteq I'$, $J \supseteq J'$ in \mathbb{R} , such that

$$\max\{\mu_{\mathbb{R}}(I) - \mu_{\mathbb{R}}(I'), \mu_{\mathbb{R}}(J) - \mu_{\mathbb{R}}(J')\} < 100\mathfrak{d}_G(A, B),$$

and $A \subseteq \chi^{-1}(I)$, $B \subseteq \chi^{-1}(J)$. □

Before state the next result, we first introduce the following definition.

Definition 7.2 (Interval progressions). We say a compact set X is an *interval progression* on G if there is a surjective continuous group homomorphism $\chi : G \rightarrow \mathbb{R}$ with compact kernel, and $X = \chi^{-1}(I + P)$, where I is a compact interval, and P is a generalized arithmetic progression.

Theorem 7.3 (Measure expansions at most 4 in noncompact groups). *Let G be a connected noncompact locally compact group, μ_G a left Haar measure, and ν_G a right Haar measure on G . Suppose A is a compact subset of G of positive measure, and*

$$(18) \quad \left(\frac{\nu_G(A)}{\nu_G(A^2)} \right)^{\frac{1}{2}} + \left(\frac{\mu_G(A)}{\mu_G(A^2)} \right)^{\frac{1}{2}} \geq 1,$$

Then G is unimodular, and one of the followings happens:

- (i) *There is a continuous surjective group homomorphism $\chi : G \rightarrow \mathbb{R}$ with compact kernel.*
 - (a) *If $\mu_G(A)/20 \leq \mathfrak{d}_G(A, A) \leq 2\mu_G(A)$, A is covered by an interval progression $\chi^{-1}(I + P)$, where $\dim P = O(1)$ and $\mu_{\mathbb{R}}(I + P) = O(1)\mu_G(A)$.*
 - (b) *If $0 < \mathfrak{d}_G(A, A) < \mu(A)/20$, then there is a compact interval $I \subseteq \mathbb{R}$, such that $A \subseteq \chi^{-1}(I)$, and*

$$\mu_G(\chi^{-1}(I)) - \mu_G(A) < 100\mathfrak{d}_G(A, A).$$

- (c) *If $\mu_G(A^2) = 2\mu_G(A)$, then there is a compact interval $I \subseteq \mathbb{R}$, such that $A = \chi^{-1}(I)$.*
- (ii) *The equality holds in (18), and there is a continuous surjective group homomorphism $\chi : G \rightarrow \widetilde{H}$ with compact kernel, where H is either \mathbb{R}^2 , or $\mathbb{R}^2 \rtimes \mathbb{T}$, or $\mathrm{PSL}_2(\mathbb{R})$, or $\mathrm{SL}_2(\mathbb{R})$. Moreover, $\chi(A)$ satisfies*

$$\mu_H(\chi(A)^2) = 4\mu_H(\chi(A))$$

where μ_H is a left Haar measure on H .

Proof. Inequality (18) implies that $\mathrm{BM}(G) \leq 2$. By Fact 3.5, we have $\mathrm{hdim}(G) \leq \mathrm{ndim}(G)/3$. As a result, either $\mathrm{BM}(G) = 1$, or $\mathrm{BM}(G) = 2$. By Theorem 3.12, there is a continuous surjective group homomorphism mapping G to a Lie group N with compact kernel, where N can only be \mathbb{R} , \mathbb{R}^2 , $\mathbb{R}^2 \rtimes \mathbb{T}$, $\mathrm{PSL}_2(\mathbb{R})$, $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, and the $ax + b$ group.

We first consider the case when $\mathrm{BM}(G) = 1$. Let $\chi : G \rightarrow \mathbb{R}$ be a continuous surjective group homomorphism with compact kernel, and let $X = \chi(A)$. Suppose that

$$\mathfrak{d}_G(A, A) < \frac{1}{20}\mu_G(A).$$

Then (i) (b) and (c) follow from Theorem 1.3. Let us now consider the case when $\mathfrak{d}_G(A, A) \geq \mu_G(A)/20$. By Proposition 5.4, $\mu_{\mathbb{R}}(X + X) \leq K\mu_{\mathbb{R}}(X)$ for some $K \leq 2^{17}$. The structural result on X follows from the fact that there are arbitrarily dense subgroups of \mathbb{R} isomorphic to \mathbb{Z} . Then one can approximate any open set by a set of the form $Y + I$ where Y is contained in some such subgroup and I is a small interval

in \mathbb{R} . The structure of Y comes from the Freiman theorem. More precisely, let $I(\varepsilon)$ be the interval $[-\varepsilon/2, \varepsilon/2]$. Then there is some $\varepsilon > 0$ such that

$$\mu_{\mathbb{R}}(X + X + I(\varepsilon)) \leq 2\mu_{\mathbb{R}}(X + X) \leq 2^{18}\mu_{\mathbb{R}}(X)$$

from the fact that $\mu_{\mathbb{R}}(X + X) = \bigcap_{\varepsilon \rightarrow 0} \mu_{\mathbb{R}}(X + X + I(\varepsilon))$. We now approximate X by a dense lattice as follows: we define

$$\mathcal{X} = X + I(\varepsilon/2) \cap \frac{\varepsilon}{10}\mathbb{Z}.$$

Then we have $X \subseteq \mathcal{X} + I(\varepsilon)$ and

$$\mathcal{X} + \mathcal{X} + I(\varepsilon/100) \subseteq X + X + I(\varepsilon).$$

Hence we have

$$|\mathcal{X} + \mathcal{X}| \frac{\varepsilon}{100} \leq \mu_{\mathbb{R}}(X + X + I(\varepsilon)) \leq 2^{18}\mu_{\mathbb{R}}(X) \leq |\mathcal{X}|2^{18}\varepsilon,$$

and hence $|\mathcal{X} + \mathcal{X}| \leq 2^{25}|\mathcal{X}|$. By Freiman's theorem, \mathcal{X} is contained in a generalized arithmetic progression P and $|P| = O(1)|\mathcal{X}|$. Therefore, we have $X \subseteq P + I(\varepsilon)$. This proves (i) (a). We remark that the argument is previously used by Tao in [24, Proposition 7.1].

Finally we assume that $\text{BM}(G) = 2$. Proposition 6.1 implies that G is unimodular. Hence N cannot be the $ax + b$ group. The desired conclusion follows from Theorem 5.1. This proves (ii). \square

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