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最小特征值有界的距离正则图

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Distance-Regular Graphs with Bounded Smallest Eigenvalues

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摘　要

在 2014 年时，Bang 证明了，对一个直径大于等于 3，顶点的度大于等于 3，围长 g 大于 3 并且满足 $g \equiv 3 \pmod 4$ 的距离正则图，当 k 足够大 (仅与围长 g 有关) 时，存在一个常数 $\gamma(g) \in (-1, -0.64)$ 使得 Γ 满足 $\theta_{\min} \geq \gamma(g)k$。在本文中，我们取 $\gamma = -\frac{D-1}{D}$。当 $D = 3, 4, 5$ 时，我们确定了所有直径为 D，度数为 k，并且最小特征值小于等于 $-\frac{D-1}{D}k$ 的非二部的距离正则图。在第二章里，我们简单介绍了一些必要的背景知识。在第三章中，我们证明了本文的主要定理。

关键词： 距离正则图，最小特征值，奇围长

ABSTRACT

In 2014, Bang showed that if Γ is a distance-regular graph with diameter $D \geq 3$, valency $k \geq 3$ and girth $g > 3$, satisfying $g \equiv 3 \pmod 4$, if k is large enough (depending only on g), then there exists $\gamma(g) \in (-1, -0.64)$ such that the smallest eigenvalue of Γ satisfies $\theta_{\text{min}} \geq \gamma(g)k$. In this paper, we fix an $\gamma = -\frac{D-1}{D}$ and try to determine all non-bipartite distance-regular graphs with diameter $D = 3, 4, 5$ with smallest eigenvalues $\theta_{\text{min}} \leq \gamma k$. In Chapter 2, we give some background in graph theory. In Chapter 3, we proof the main theorem.

Keywords: Distance-regular graph, smallest eigenvalue, odd girth

Chapter 1 Introduction

Distance-regular graphs are graphs having lots of combinatorial symmetry, that means that given an arbitrary ordered pair of vertices u, v with $d(u, v) = h$, the number of vertices that are at distance i from u and distance j from v is a constant. It does not depend on the chosen pair u, v, only depends on h, i and j. Biggs [11] introduced distance-regular graphs in 1974, by observing several combinatorial and algebraic properties of distance-transitive graphs were holding for this wider class of graphs. Distance-regular graphs have applications in several fields, for example, Hamming graphs and Johnson graphs link to coding theory and design theory, respectively. There are many more interesting links to other fields, such as finite group theory, finite geometry, representation theory, and orthogonal polynomials. In graph theory, distance-regular graphs are always used as test instances for problems for general graphs or other combinatorial structures. Distance-regular graphs also have applications in quantum information theory, diffusion models, networks, and even finance.

The spectrum of a distance-regular graph contains quite some information about the graph, it has many useful applications. In this paper, we focus on the spectrum of distance-regular graphs.

Bang [1] showed that if Γ is a distance-regular graph having diameter $D \geq 3$, valency $k \geq 3$ and girth $g > 3$ satisfying $g \equiv 3 \pmod 4$, when k is not too small (depending only on g), there exists $\gamma(g) \in (-1, -0.64)$ such that the smallest eigenvalue of Γ satisfies $\theta_{\text{min}} \geq \gamma(g)k$. In this thesis, we fix an $\gamma = -\frac{D-1}{D}$ and try to determine all non-bipartite distance-regular graphs with diameter $D = 3, 4, 5$ and smallest eigenvalues $\theta_{\text{min}} \leq \gamma k$. The main result in the thesis is contained in an ongoing project [12].

Theorem 1.0.1. *Let Γ be a non-bipartite distance-regular graph with valency k, diameter D and smallest eigenvalue $\theta_{\text{min}} \leq -\frac{D-1}{D}k$.*

1. If $D = 3$, then Γ is one of the following:
(a) *the 7-gon with intersection array $\{2, 1, 1; 1, 1, 1\}$;*
(b) *the Odd graph O_4 with intersection array $\{4, 3, 3; 1, 1, 2\}$;*
(c) *the folded 7-cube with intersection array $\{7, 6, 5; 1, 2, 3\}$.*

2. If $D = 4$, then Γ is one of the following:
(a) *the Coxeter graph with intersection array $\{3, 2, 2, 1; 1, 1, 1, 2\}$;*

(b) *the* 9*-gon with intersection array* $\{2,1,1,1;1,1,1,1\}$*;*
(c) *the Odd graph* O_5 *with intersection array* $\{5,4,4,3;1,1,2,2\}$*;*
(d) *the folded* 9*-cube with intersection array* $\{9,8,7,6;1,2,3,4\}$.

3. If $D=5$*, then* Γ *is one of the following:*
(a) *the* 11*-gon with intersection array* $\{2,1,1,1,1;1,1,1,1,1\}$*;*
(b) *the Odd graph* O_6 *with intersection array* $\{6,5,5,4,4;1,1,2,2,3\}$*;*
(c) *the folded* 11*-cube with intersection array* $\{11,10,9,8,7;1,2,3,4,5\}$.

This paper is organized as follows: In Chapter 2, we give the basic definitions and concepts in graph theory, including basic properties of distance-regular graphs and the matrix theory. In Chapter 3, we give the proof of the theorem above.

Chapter 2 Preliminaries

2.1 Graphs

All the graphs considered in this paper are finite, undirected and simple. A $graph$ is a pair $\Gamma = (V, E)$ consisting of a vertex set V and an edge set E, referred to as the $edge\ set$ of Γ, where an edge is an unordered pair of distinct vertices of Γ. We usually use xy to denote an edge, and we say that x and y are $adjacent$ or y is a $neighbor$ of x, use the notation $x \sim y$. A 2-subset of V not in E is called a $nonedge$ of Γ, and the $complement$ of Γ, often denoted $\bar{\Gamma}$, is the graph with vertex set V whose edges are all the nonedges of Γ. The $distance$ in the graph between two vertices x and y is denoted by $d(x, y) = d_\Gamma(x, y)$, and is given by the length of the shortest path between x and y in Γ. The $diameter$ of the graph is $D = D(\Gamma) = \max_{x,y\in V} d(x, y)$. The set of vertices at distance i from a given vertex $x \in V$ is denoted by $\Gamma_i(x)$, for $i = 0, 1, \ldots, D$, and let $\Gamma(x) = \Gamma_1(x)$ for the convenience. A $path$ of length p from x to y in a graph is a sequence of $p + 1$ distinct vertices starting with x and ending with y such that consecutive vertices are adjacent. If there exist a path between any two vertices in Γ, we say the graph Γ is $connected$, otherwise $disconnected$. A $walk$ of length t in Γ is a sequence of vertices $v_0 \sim v_1 \sim \cdots \sim v_t$. Note that the important difference between walk and path is that a walk is permitted to use vertices more than once.

A graph Γ is called $complete$ or $clique$ when any two of its vertices are adjacent. The complete graph on n vertices is denoted by K_n. A $coclique$ is a graph in which no two vertices are adjacent.The $valency$ or $degree$ $k(x)$ of a vertex x is the cardinality of the neighbors of x. In particular, Γ is called $regular$ with valency k if $k = |\Gamma(x)|$ holds for all vertices $x \in V(\Gamma)$.

Two graphs G and H are equal or $isomorphic$ if there is a bijection φ from $V(G)$ to $V(H)$, such that $x \sim y$ in G if and only of $\varphi(x) \sim \varphi(y)$ in H. A $subgraph$ of a graph G is a graph H, where $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(G) = V(H)$, we call H a $spanning\ subgraph$ of G. A subgraph H of G is an $induced\ subgraph$ if any two vertices of $V(H)$ are adjacent in H if and only if they are adjacent in G. A $cycle$ is a connected graph and every vertex has exactly two neighbors. A cycle in a graph refers to an induced subgraph of Γ that is a cycle.

The $girth$ of Γ, denoted by $g = g(\Gamma)$, is the length of the shortest cycle in Γ. The $odd\ girth$ of Γ is the length of the shortest odd cycle in Γ. A subgraph H of Γ is called

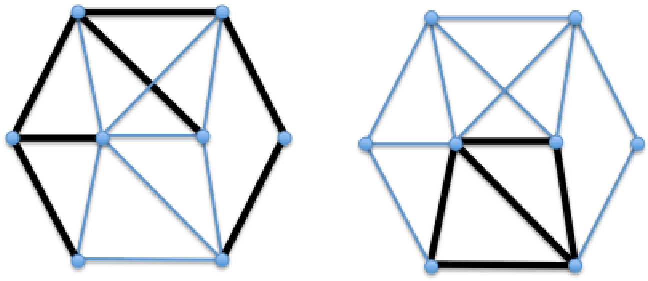

Figure 2.1 A spanning subgraph and an induced subgraph of a graph

isometric if $d_\Gamma(x,y) = d_H(x,y)$ for all $x, y \in V(H)$. Note that an isometric subgraph is always induced. Let H be an induced subgraph of Γ, the *width* of H, denoted by $w = w(H)$, is $w(H) = \max\{d_\Gamma(x,y) \mid x, y \in V(H)\}$. Given a vertex $x \in V(\Gamma)$, the *local graph* $\Delta(x)$ of x is the induced subgraph on the vertex set $\Gamma(x)$.

2.2 Distance-Regular Graphs

2.2.1 Definitions and Properties

For any two vertices u, v at distance i, we consider the numbers $c_i(u,v) = |\Gamma_{i-1}(u) \cap \Gamma(v)|$, $a_i(u,v) = |\Gamma_i(u) \cap \Gamma(v)|$ and $b_i(u,v) = |\Gamma_{i+1}(u) \cap \Gamma(v)|$. A connected graph Γ with diameter D is called *distance-regular* if a_i, b_i and c_i are constants for $i = 0, 1, 2, \ldots, D$, that means, $a_i(u,v), b_i(u,v)$ and $c_i(u,v)$ depend only on $i = d_\Gamma(u,v)$ not on the choice of vertices (u,v) with $d(u,v) = i$. We call a_i, b_i and c_i the *intersection numbers*. Set $c_0 = b_D = 0$, obviously we have $a_0 = 0$ and $c_1 = 1$. It follows that Γ is a regular graph with valency $k = b_0$ and that $a_i + b_i + c_i = k$ for all $i = 0, 1, \ldots, D$. Since a_i can be expressed in terms of the others, the *intersection array* of a distance-regular graph with diameter D is the array $\{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}$. Note that every vertex has a constant number of vertices k_i at given distance i, that is, $k_i = |\Gamma_i(x)|$ for all $x \in V$. Counting the number of edges between $\Gamma_i(x)$ and $\Gamma_{i+1}(x)$ in two ways, we have $k_0 = 1$ and $k_{i+1} = b_i k_i / c_{i+1}$ for all $i = 0, 1, \ldots, D-1$. The number of vertices now follows as $|V| = k_0 + k_1 + \cdots + k_D$. In particularly, a distance-regular graph with diameter $D = 2$ is *strongly-regular* with parameters (v, k, a_1, c_2).

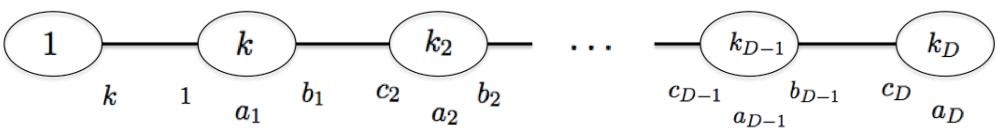

Figure 2.2 A distance-regular graph

A connected graph Γ with diameter $D \geq t$ for a positive integer t is called a *t-partially*

$diatance$-$regular\ graph$ if there exist intersection numbers a_i, b_i, c_i for all $i =\leq t$. In [10], Fiol and Garriga introduced t-walk regular graphs as a generalization of distance-regular graphs. For an integer $t \in (0, D]$, Γ with diameter D is called a t-$walk\ regular$ graph if for any vertices $x, y \in V(\Gamma)$ with $d_\Gamma(x, y) \leq t$, the number of walks of any given length between x and y only depends on $d_\Gamma(x, y)$. In [7, Proposition 3.15] we see that a t-walk regular graph is a t-partially distance-regular graph with intersection numbers $b_0, b_1, \ldots, b_t, c_1, c_2, \ldots, c_t$. In particularly, a distance-regular graph with diameter D is D-walk regular.

2.2.2 Examples

The complete graphs. The complete graphs K_v is a graph where all vertices are adjacent to each other. Obviously they are distance-regular graph with diameter $D = 1$ with intersection array $\{v - 1; 1\}$.

The polygons. The polygons v-gon are the distance-regular graphs with diameter $D = [\frac{v-1}{2}]$ and valency 2, where $[\cdot]$ is Gaussian function. They have intersection array $\{2, 1, 1, \ldots, 1; 1, 1, \ldots, 1\}$ if v is odd, and $\{2, 1, \ldots, 1; 1, \ldots, 1, 2\}$ if v is even.

Figure 2.3 Polygon

The Odd graphs. For an integer $t \geq 2$, the vertices of the Odd graph O_t are the $(t-1)$-subsets of a set of size $2t-1$. Two vertices are adjacent if the corresponding subsets are disjoint. The Odd graph O_t is distance-regular with diameter $t - 1$. For $t = 2l - 1$, the intersection array is $\{k, k-1, k-1, \ldots, l+1, l+1, l; 1, 1, 2, 2, \ldots, l-1, l-1\}$, and for $t = 2l$, the intersection array is $\{k, k-1, k-1, \ldots, l+1, l+1; 1, 1, 2, 2, \ldots, l-1, l-1, l\}$. Obviously, in the Odd graphs, the intersection numbers a_i are zero for all $i = 0, 1, \ldots, D-1$, but $a_D = l$. One of the famous examples of Odd graph is the Peterson graph O_3.

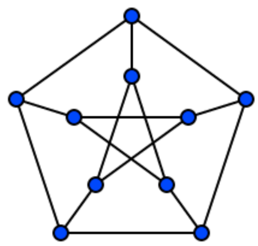

Figure 2.4 The Peterson graph

The folded cubes. The folded n-cube is a partition graph, it can be described as that the graph whose vertices are the partitions of an n-set into two subsets. Two partitions being adjacent when their common refinement contains a set of size one. For $n \geq 3$, the intersection array is given by diameter $D = [n/2]$, and the intersection numbers $b_j = n - j$ and $c_j = j$. If n is even, then $c_D = n$. The eigenvalues and multiplicities are $\theta_j = n - 4j$, $m(\theta_j) = \binom{n}{2j}$.

2.3 Matrix Theory

The $adjacency\ matrix$ A of a graph Γ is the $v \times v$ symmetric matrix indexed by the vertices of Γ, whose entries $a_{\gamma\delta}$ are given by $a_{\gamma\delta} = 1$ if $\gamma \sim \delta$, and $a_{\gamma\delta} = 0$ otherwise. Since A is real and symmetric, its eigenvalues are real numbers, they are called the $eigenvalue$ of Γ. If Γ is regular of valency k, its adjacency matrix A satisfies the equation $AJ = kJ$ and $A\mathbf{1} = k\mathbf{1}$. In particular, k is an eigenvalue of Γ. In [11, Chapter 8.6], we also know that the smallest eigenvalue of Γ is at least $-k$, and the eigenvalue of the induced subgraph of Γ is controlled by the eigenvalue of Γ.

Lemma 2.3.1. *Let Y be an induced subgraph of X. Then*

$$\theta_{\text{min}}(X) \leq \theta_{\text{min}}(Y) \leq \theta_{\text{max}}(Y) \leq \theta_{\text{max}}(X)$$

The $adjacency\ algebra$ of Γ, denoted by $\mathbb{A} = \mathbb{A}(\Gamma)$ and $\mathbb{A} = \mathbb{R}[A]$. In [3, Lemma 2.5], we see

Lemma 2.3.2. *The number of walks of length l in Γ, joining v_i to v_j, is the entry in position (i, j) of the matrix A^l.*

Using this, we can see the relation between the number of distinct eigenvalues and the diameter of the graph. Assume first that Γ has distinct eigenvalues $\theta_0, \theta_1, \ldots, \theta_d$. Because the minimal polynomial of A has degree $d+1$, it is clear that $\{I, A, A^2, \ldots, A^d\}$ is a basis of $\mathbb{A}$, hence dim $\mathbb{A} = d + 1$.

Now we consider the case that Γ is distance-regular. The adjacency matrix A_i of Γ_i is called the $distance\text{-}i\ matrix$ of Γ, for $i = 0, 1, \ldots, D$. By Lemma 2.3.2, we obtain the equation

$$AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1} \tag{2.1}$$

for $i = 0, 1, \ldots, D$. Set $b_{-1}A_{-1} = c_{D+1}A_{D+1} = 0$, then there exist polynomials p_i of degree i such that $A_i = p_i(A)$. Hence $\{I = A_0, A = A_1, A_2, \ldots, A_D\}$ is also a basis of $\mathbb{A}$. We may conclude the following:

Proposition 2.3.1. *Let Γ be a distance-regular graph with diameter D. Then* dim $\mathbb{A} = D + 1$, *and Γ has exactly $D + 1$ distinct eigenvalues.*

In [8, Proposition 2.7] we can see that the $D+1$ distinct eigenvalues of Γ with diameter D are the eigenvalues of the $intersection\ matrix$:

$$L = \begin{bmatrix} 0 & b_0 & & & & \\ c_1 & a_1 & b_1 & & 0 & \\ & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & 0 & & \cdot & \cdot & b_{D-1} \\ & & & & c_D & a_D \end{bmatrix}$$

Let θ be an eigenvalue of L with corresponding right eigenvector $\mathbf{u} = (u_0, u_1, \ldots, u_D)^\top$, then we have $L\mathbf{u} = \theta\mathbf{u}$, where $u_0 = 1$, $u_1 = \theta/k$, and

$$c_i u_{i-1} + a_i u_i + b_i u_{i+1} = \theta u_i \tag{2.2}$$

for $i = 1, 2, \ldots, D$. The sequence $(u_i)_{i=0}^D$ is called the $standard\ sequence$ of Γ for the eigenvalue θ.

For an eigenvalue θ of Γ, the $multiplicity$ of θ is denoted by $m(\theta) = m_A(\theta)$. Let $\{v_1, v_2, \ldots, v_m\}$ be a set of orthonormal eigenvectors of A corresponding to θ, then let W be a matrix whose columns are v_i, where $1 \leq i \leq m$. The matrix $E_\theta = WW^T$ is called a $minimal\ idempotent$ corresponding to θ. For $i = 0, 1, \ldots, D$, we define the matrix $E_i = \Pi_{j=0, j\neq i}^{D} \frac{A-\theta_j I}{\theta_i - \theta_j}$, then E_θ is one of the matrices E_i. We shall see the set $\{E_0, E_1, \ldots, E_D\}$ forms another basis of $\mathbb{A}$. Indeed, let $\mathbf{v}$ be an eigenvector of θ_j, then $E_i\mathbf{v} = \delta_{ij}\mathbf{v}$. That means that $\{E_0, E_1, \ldots, E_D\}$ forms a linearly independent set of matrices in $\mathbb{A}$. Using this, by [8, Theorem 2.8], we get the relation between the multiplicities of the eigenvalues of Γ and the intersection numbers. This is known as *Biggs' Formula*.

Theorem 2.3.1. (Biggs' Formula) *Let Γ be a distance-regular graph with diameter D and v vertices. Let θ be an eigenvalue of Γ and $(u_i)_{i=0}^D$ be the standard sequence with respect to θ. Then the multiplicity $m(\theta)$ satisfies*

$$m(\theta) = \frac{v}{\sum_{i=0}^{D} k_i u_i^2} \tag{2.3}$$

A clique C with $1 - k/\theta_{\text{min}}$ vertices is called a $Delsarte\ clique$ of Γ. The following result was first shown by Delsarte [9] for strongly-regular graphs and then extended by Godsil to the class of distance-regular graphs.

Theorem 2.3.2. (Delsarte-Godsil Bound) *Let Γ be a distance-regular graph with valency $k \geq 2$, diameter $D \geq 2$ and smallest eigenvalue θ_{min}. Let $C \subseteq V(\Gamma)$ be a clique* *with* c *vertices. Then*

$$c \le 1 + \frac{k}{-\theta_{\text{min}}} \tag{2.4}$$

with equality if and only if C *is a completely regular code with covering radius* $D - 1$.

Chapter 3 Proof of the Main Theorem

3.1 Distance-regular graphs with odd girth 7 and 9

Assume Γ is a distance-regular graph with diameter D and odd girth $g = 2t+1$. Let $(u_i)_{i=0}^{D}$ be the standard sequence for the smallest eigenvalue $\theta_{\min}$. Let Δ be a g-gon and let η be an eigenvalue of Δ. Because Δ is an isometric subgraph of Γ, by [2, Proposition 3.1], we have

$$\sum_{i=0}^{t} k_i(\Delta)u_i \geq 0 \tag{3.1}$$

$$\sum_{i=0}^{t} p_i(\eta)u_i \geq 0 \tag{3.2}$$

By equation 2.1, we see that $p_i(x)$ are the polynomials defined as the following:

$$\begin{aligned}
&p_0(x) = 1\\
&p_1(x) = x\\
&p_2(x) = x^2 - 2\\
&p_i(x) = xp_{i-1}(x) - p_{i-2}(x) \quad (3 \leq i \leq t)
\end{aligned}$$

When $g = 5$, by Equation 3.1, $u_0 + 2u_1 + 2u_2 + 2u_3 \geq 0$. Let $\varepsilon = \theta_{\min}/k$, applying Equation 2.2, we have

$$1 + 2\varepsilon + 2\frac{\varepsilon^2 k - 1}{k-1} + 2\frac{\varepsilon(\varepsilon^2 k^2 - (1+c_2)k + c_2)}{(k-1)(k-c_2)} \geq 0 \tag{3.3}$$

When $g = 9$, we can easily get that

Lemma 3.1.1. *Let Γ be a distance-regular graph with diameter $D \geq 2$, valency $k \geq 2$ and odd girth 9. Let $\theta_{\min} = \varepsilon k$ be the smallest eigenvalue of Γ. If $\varepsilon < -0.755$, then k is less than the largest root of the polynomial*

$$(\varepsilon^4 - 2\varepsilon^3 + \varepsilon^2 + \varepsilon + 1)k^3 + (2\varepsilon^3 - 5\varepsilon^2 + 3\varepsilon + 1)k^2 + (10\varepsilon^2 - 8\varepsilon - 2)k - 1 + \varepsilon = 0 \tag{3.4}$$

Proof. When $g = 9$, by Equation 3.2, take $\eta = -1$, we have $u_0 - u_1 - u_2 + 2u_3 - u_4 \geq 0$. Since $\varepsilon < -0.755 < -\frac{3}{4}$, by [13, Proposition 5.1], consider the induced K_{2,c_2}, we have that $\frac{4c_2}{2+c_2} \leq 1 - \frac{k-1}{\theta_{\min}-1} < \frac{7}{3}$, $c_2 \leq 2$. Also, the coefficient of k^3 is negative, then we can estimate $u_0 - u_1 - u_2 + 2u_3 - u_4$ by $c_2 \leq 2$ and $c_3 \in [2,k]$, and easily get the result. $\square$

3.2 $D = 3, 4$

The following proposition is a direct result of [13, Theorem 1.2].

Proposition 3.2.1. *Let Γ be a non-bipartite distance-regular graph with valency k, diameter $D = 3$ and smallest eigenvalue $\theta_{\min} \le -\frac{2}{3}k$. Then Γ is one of the following:*
(a) *the 7-gon with intersection array $\{2, 1, 1; 1, 1, 1\}$;*
(b) *the Odd graph O_4 with intersection array $\{4, 3, 3; 1, 1, 2\}$;*
(c) *the folded 7-cube with intersection array $\{7, 6, 5; 1, 2, 3\}$.*

Then we consider the case $D = 4$.

Proposition 3.2.2. *Let Γ be a non-bipartite distance-regular graph with valency k, diameter $D = 4$ and smallest eigenvalue $\theta_{\min} \le -\frac{3}{4}k$. Then Γ is one of the following:*
(a) *the Coxeter graph with intersection array $\{3, 2, 2, 1; 1, 1, 1, 2\}$;*
(b) *the 9-gon with intersection array $\{2, 1, 1, 1; 1, 1, 1, 1\}$;*
(c) *the Odd graph O_5 with intersection array $\{5, 4, 4, 3; 1, 1, 2, 2\}$;*
(d) *the folded 9-cube with intersection array $\{9, 8, 7, 6; 1, 2, 3, 4\}$.*

Proof. By [13, Proposition 5.1], consider the induced K_{2,c_2}, we have that $\frac{4c_2}{2+c_2} \le 1 - \frac{k-1}{\theta_{min}-1} < \frac{7}{3}$, $c_2 \le 2$. By Theorem 2.3.2, the size of the maximal clique is at most $1 - \frac{k}{\theta_{\min}} < 3$, that means $a_1 = 0$.

Assume that $a_2 \neq 0$, Γ must contain an induced 5-gon. Then by [2, Corollary 3.4], as $-\frac{1}{2} + \frac{1}{2}\sqrt{5}$ is the second large eigenvalue of the pentagon, $\theta_{\min} \ge -1 - \frac{2(k-1)}{1+\sqrt{5}}$. Since $\theta_{\min} \le -\frac{3}{4}k$, we see $k \le 2$.

Hence we have $a_1 = a_2 = 0$. Assume $a_3 \neq 0$, Γ must contain an induced 7-gon. By Equation 3.3, applying $\varepsilon = -\frac{3}{4}$ and $c_2 = 1$ or 2. Therefore,

$$-\frac{1}{2} + \frac{\frac{9}{8}k - 2}{k-1} - \frac{\frac{3}{2}(\frac{9}{16}k^2 - (1+c_2)k + c_2)}{(k-1)(k-c_2)} \ge 0 \tag{3.5}$$

Then we obtain

$$\begin{cases} k \le 4 & c_2 = 1 \\ k \le 8 & c_2 = 2 \end{cases}$$

By the method in [5, Proposition 4.16], we check all the feasible intersection arrays with $c_2 = 1$ or 2, $a_1 = a_2 = 0$, $a_3 \neq 0$ and $k \le 8$, we see that is the Coxeter graph by [5, Theorem 12.3.1].

Now we consider the case $a_1 = a_2 = a_3 = 0$. Assume that $k \ge 38$, then $|u_2| \ge \frac{\frac{9}{16}k-1}{k-1} \ge$

0.5506. Consider the intersection matrix

$$L = \begin{bmatrix} 0 & k & 0 & 0 & 0 \\ 1 & 0 & k-1 & 0 & 0 \\ 0 & c_2 & 0 & k-c_2 & 0 \\ 0 & 0 & c_3 & 0 & k-c_3 \\ 0 & 0 & 0 & c_4 & k-c_4 \end{bmatrix}$$

we have

$$\begin{aligned} k^2 + \theta_{\min}^2 &\leq tr(L^2) \\ &\leq k^2 + 6k + c_4^2 + 2c_3(k - c_4) \\ &\leq k^2 + 6k + c_4(2k - c_4). \end{aligned}$$

This implies that $c_4 \geq 0.2283k$. As

$$\begin{aligned} |u_3| &= |\frac{\theta u_2 - c_2 u_1}{k - c_2}| \\ &\geq \frac{|\theta u_2| - c_2}{k - c_2} \\ &\geq 0.3803, \end{aligned}$$

Now by Theorem 2.3.1, we have $m = \frac{v}{\sum_{i=0}^{4} k_i u_i^2}$, where v is the number of vertices in Γ. Note that, for positive real number a, b, c, d, if $a/c \geq b/d$, then

$$\frac{a}{c} \geq \frac{a+b}{c+d} \geq \frac{b}{d}$$

holds. Using this, we have that

$$\begin{aligned} m &\leq \frac{k_3 + k_4}{k_3 u_3^2} \\ &\leq \frac{1 + \frac{k}{c_4}}{u_3^2} < 38 \end{aligned}$$

By [2, Proposition 3.3], we see $k \leq m \leq 37$.

Then we check all the feasible intersection arrays with $c_2 = 1$ or 2, $a_1 = a_2 = a_3 = 0$, and $k \leq 37$, we get the rest three graphs. □

3.3 $D = 5$

Proposition 3.3.1. *Let Γ be a non-bipartite distance-regular graph with valency k, diameter $D = 5$ and smallest eigenvalue $\theta_{\min} \leq -\frac{4}{5}k$. Then Γ is one of the following:*
(a) *the 11-gon with intersection array $\{2,1,1,1,1;1,1,1,1,1\}$;*
(b) *the Odd graph O_6 with intersection array $\{6,5,5,4,4;1,1,2,2,3\}$;*
(c) *the folded 11-cube with intersection array $\{11,10,9,8,7;1,2,3,4,5\}$.*

Proof. Similarly, by [13, Proposition 5.1], we have $c_2 \leq 2$, and by Theorem 2.3.2, we have $a_1 = 0$. We may assume $k \geq 5$, otherwise Γ is the 11-gon by [4] and [6, Theorem 1.1].

If $a_2 \neq 0$, Γ must contain an induced 5-gon. For similar reasons, $\theta_{\text{min}} \geq -1 - \frac{2(k-1)}{1+\sqrt{5}}$, and $k \leq 2$.

Hence we have $a_1 = a_2 = 0$. If $a_3 \neq 0$, Γ must contain an induced 7-gon. Then by Equation 3.3, we have

$$\begin{cases} k \leq 3 & c_2 = 1 \\ k \leq 5 & c_2 = 2 \end{cases}$$

For second case, when $c_2 = 2$, by [5, Theorem 1.13.2], Γ is the 5-cube, which is bipartite.

Then we consider the case $a_1 = a_2 = a_3 = 0$. Assume $a_4 \neq 0$, the odd girth of Γ must be 9. Then by Lemma 3.1.1, let $\varepsilon = -\frac{4}{5}$, we get $k \leq 18$. Then we check all the feasible intersection arrays with $a_1 = a_2 = a_3 = 0$, $c_2 = 1$ or 2 and $k \leq 18$, no such array exists.

Now we consider the case $a_1 = a_2 = a_3 = a_4 = 0$. Assume that $k \geq 61$. Also consider the intersection matrix

$$L = \begin{bmatrix} 0 & k & 0 & 0 & 0 & 0 \\ 1 & 0 & k-1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & k-c_2 & 0 & 0 \\ 0 & 0 & c_3 & 0 & k-c_3 & 0 \\ 0 & 0 & 0 & c_4 & 0 & k-c_4 \\ 0 & 0 & 0 & 0 & c_5 & k-c_5 \end{bmatrix}$$

we have

$$\begin{aligned} k^2 + \theta_{\text{min}}^2 &\leq tr(L^2) \\ &\leq k^2 + 6k + c_5^2 + 2c_4(2k - c_5) \\ &\leq k^2 + 6k + 4c_5k - c_5^2 \end{aligned}$$

This implies that $c_5 \geq 0.1403k$. As

$$\begin{aligned} |u_3| &= |\frac{\theta_{\text{min}}(\theta_{\text{min}}^2 - (1+c_2)k + c_2)}{k(k-1)(k-2)}| \\ &\geq 0.4904, \end{aligned}$$

In this situation, we may assume that $c_4 = \alpha k$ and $c_5 = \beta k$. Then by Equation 2.2, we have

$$\begin{aligned} u_4 &= \frac{\theta_{\text{min}} u_3 - c_4 u_2}{k - c_4} \\ &\geq \frac{\frac{4}{5}|u_3| - \alpha}{1 - \alpha} \end{aligned}$$

Therefore,

$$
\begin{aligned}
m &\le \frac{k_3+k_4+k_5}{k_3u_3^2+k_4u_4^2}\\
&\le \frac{1+\frac{k}{c_4}(1+\frac{k-c_4}{c_5})}{u_3^2+\frac{k}{c_4}u_4^2}\\
&= \frac{1+\frac{1}{\alpha}(1+\frac{1-\alpha}{\beta})}{u_3^2+\frac{1}{\alpha}(\frac{\frac{4}{5}|u_3|-\alpha}{1-\alpha})^2}
\end{aligned}
$$

Then use the fact $0<\alpha\le\beta$ and $0.1403\le\beta\le 1$, we get $m<61$. By [2, Proposition 3.3], we get $k<61$, contradiction.

It follows that $k\le 60$. Then we check all feasible intersection arrays with $a_1=a_2=a_3=a_4=0$, $c_2=1$ or 2, and $\theta_{\text{min}}\le -\frac{4}{5}k$, the folded 11-cube and the odd graph O_6 are the only possible ones.

This completes the proof. □

Now Theorem 1.0.1 follows from Proposition 3.2.1, Proposition 3.2.2 and Proposition 3.3.1.

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